Capacity Penalty due to Ideal Zero-Forcing Decision-Feedback Equalization

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Abstract — We compare the capacity C_{ZF} of an idealized zero-forcing decision-feedback equalization (ZF-DFE) system to the capacity C of the underlying Gaussian-noise channel. We find that, for strictly bandlimited channels, the capacity penalty gets vanishingly small as the signal power increases. For non-strictly bandlimited channels, on the other hand, the capacity difference $C - C_{ZF}$ grows without bound as the signal power increases, and the asymptotic ratio C_{ZF} / C is as low as 93.6% for some channels. We conclude that ZF-DFE on non-strictly bandlimited channels is a suboptimal, information-lossy equalization technique, even in the limit of infinite signal power.

1. INTRODUCTION

Equalizers are used in digital communication systems because they make it easier for the receiver to recover the transmitted data when the intervening channel induces temporal dispersion. In the process, some equalizers destroy information and hence cause a reduction in Shannon capacity. A zero-forcing decision-feedback equalizer (ZF-DFE) uses past decisions to subtract off postcursor intersymbol interference (ISI) [1]. Neglecting error propagation, this results in an ISI-free additive white Gaussian-noise channel, allowing coded modulation schemes to be used effectively. The question naturally arises: relative to the underlying continuous-time channel, what is the loss in capacity incurred by the use of ZF-DFE? In this paper, we quantify this loss in two ways: with a power penalty Γ and a capacity penalty ΔC.

We define the power penalty in a way similar to Price, who showed that, for a linear Gaussian-noise channel with input power P and capacity $C(P)$, an uncoded PAM system using ZF-DFE with a bit rate of $C(P)$ can achieve a 10^{-5} error rate when the average transmitter power is Γ times larger than P, where Γ is around 8 dB for most channels [2]. Instead of restricting attention to a particular modulation scheme and error rate, we extend this result by finding the extra power Γ required for the capacity C_{ZF} of the ideal ZF-DFE channel to equal the capacity of the underlying channel. In other words, we solve the equation $C_{ZF}(\Gamma P) = C(P)$ for Γ.

This power penalty, although a useful quantity, does not characterize the capacity difference $\Delta C \equiv C - C_{ZF}$. For a given power penalty Γ, ΔC can approach either zero, a constant, or infinity in the limit of large signal power, depending on the channel. Therefore, to fully understand the effects of ZF-DFE on capacity, we must consider both Γ and ΔC.

In the next section we relate the water-pouring method for calculating capacity to the arithmetic and geometric means of the inverse folded spectrum. In Sect. 3 we show that, in the limit of large signal power, the capacity of a loosely (non-strictly) bandlimited channel is asymptotically proportional to the *asymptotic order* of the channel. In Sect. 4 we develop a model for the ideal ZF-DFE channel and quantify the power penalty and capacity penalty due to ideal ZF-DFE.

Although our results are framed in the context of ZF-DFE, they are also applicable to systems using precoding [3][4].

2. WATER POURING

In this paper we consider real-valued linear Gaussian-noise channels described by:

$$y(t) = x(t) \otimes b(t) + n(t) \tag{1}$$

where $y(t)$ is the received signal, $x(t)$ is the transmitted signal, $b(t)$ is the channel impulse response with Fourier transform $B(\omega)$, the symbol $\otimes$ denotes convolution, and $n(t)$ is white Gaussian noise with power spectral density (PSD) N_0. We restrict our attention to baseband channels for simplicity; extensions to passband channels are straightforward [5].

In this context, we define a *bowl*:

$$\beta(\omega) \equiv |B(\omega)|^{-2}, \tag{2}$$

its arithmetic mean over a band of width Ω:

$$\mathcal{A} \equiv \frac{1}{\Omega} \int_0^{\Omega} \beta(\omega)\, d\omega\,, \tag{3}$$

and its geometric mean over a band of width Ω:

$$\mathcal{G} \equiv \exp\{ \frac{1}{\Omega} \int_0^{\Omega} \log \beta(\omega)\, d\omega \}\,. \tag{4}$$

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The integral inequality $\int \log\beta(x)\, dx \le \log \int \beta(x)\, dx$ implies that $\mathcal{G} \le \mathcal{A}$, with equality only when $\beta(\omega)$ is a constant.

The capacity of the channel in (1), subject to a signal power constraint of $\mathrm{E}[x(t)^2] \le P$, can be calculated using the water-pouring method [6][7], which we summarize here in terms of $\mathcal{A}$ and $\mathcal{G}$:

Step 1. Create a bowl by plotting $\beta(\omega)$ versus ω.

Step 2. From the power constraint P, determine the water-pouring bandwidth $\Omega = \pi / T$ by pouring water into the bowl until the normalized area of the water equals the normalized power constraint:

$$\begin{aligned} PT/N_0 &= \frac{1}{\Omega} \int_{\Omega} [\mathcal{L} - \beta(\omega)]\, d\omega \\ &= \mathcal{L} - \mathcal{A}. \end{aligned} \tag{5}$$

Step 3. Using (3) and (4), calculate the capacity (in b/s) using:

$$C = \frac{1}{2T} \log_2(\mathcal{L}/\mathcal{G}) \tag{6}$$

$$= \frac{1}{2T} \log_2 \left[\frac{\mathcal{A} + PT/N_0}{\mathcal{G}} \right]. \tag{7}$$

The integral in (5) is over the set of frequencies for which the "water level" $\mathcal{L}$ satisfies $\mathcal{L} \ge \beta(\omega)$. In the general case this set will consist of a union of disjoint intervals, but for simplicity we assume it takes the form $[0, \Omega]$. Extensions to the general case are straightforward [5]. We define $T \equiv \pi/\Omega$ as the Nyquist interval corresponding to the bandwidth Ω.[1] The input signal that achieves capacity is a Gaussian random process with PSD proportional to the difference between the water level and the bowl [6][7]:

$$S_x(\omega) = N_0 \max[0, \mathcal{L} - \beta(\omega)]. \tag{8}$$

The term PT/N_0 in (7) has two useful interpretations: first, because N_0/T is the noise power within the transmission band, PT/N_0 is the signal-to-noise ratio; and second, being the difference between the water level and the average height of the bowl (see (5)), PT/N_0 is the average water depth.

We see that capacity is intimately related to the arithmetic and geometric means; for a given power constraint and corresponding water-pouring bandwidth, channels with identical $\mathcal{A}$ and $\mathcal{G}$ have the same capacity, regardless of the particular shape of their transfer functions. Note that for flat (ISI-free) channels, $\mathcal{A} = \mathcal{G} = 1$, and (7) reduces to the well-known formula for the capacity of an ideally bandlimited Gaussian-noise channel.

1. We will use Ω and π/T synonymously throughout this paper.

3. ASYMPTOTIC ORDER and CAPACITY AT HIGH SIGNAL POWERS

We first define the *asymptotic order* of a channel, a key parameter which reappears throughout the paper, and then relate it to capacity in the limit of large signal power.

3.1 Asymptotic Order

For a differentiable bowl $\beta : [0, \infty) \to (0, \infty)$ satisfying $\beta(\omega) \to \infty$ as $\omega \to \infty$, define its *asymptotic order* γ by the following limit, when it exists:

$$\gamma \equiv \lim_{\omega \to \infty} \frac{\log \beta(\omega)}{\log \omega} = \lim_{\omega \to \infty} \frac{\omega \beta'(\omega)}{\beta(\omega)}. \tag{9}$$

The limit may not exist (consider $\beta(\omega) = \omega^{3 + \cos(\omega)}$). It can be shown that β eventually increasing is a necessary but not sufficient condition for γ to exist.

The parameter $\gamma \in [0, \infty) \cup \{\infty\}$ is a generalization of the order of a polynomial; when $\beta(\omega)$ is a polynomial of order n, γ is just n. Other examples include $\gamma = 0$ for $\beta(\omega) = \log(2 + \omega)$, and $\gamma = \infty$ for $\beta(\omega) = e^{\omega}$. A bowl with order γ will behave like a polynomial of order γ at high frequencies. Another interpretation of γ is that 10γ is the asymptotic roll-off of the channel in dB per decade.

In this paper we restrict our attention to bowls that satisfy the following mild conditions.

Assumption 1. Assume $\beta : [0, \infty) \to (0, \infty)$ is eventually differentiable,[2] eventually unbounded, and its asymptotic order γ exists.

These include, for example, channels with proper, rational transfer functions.

We are particularly interested in the behavior of capacity at high signal powers, which from (6) is governed by the behavior of the ratio $\mathcal{G}/\mathcal{L}$. In Sect. 4, when we examine the capacity penalty due to ideal ZF-DFE, the ratio $\mathcal{A}/\mathcal{L}$ becomes important as well. The following key result relates the limit of these ratios to the asymptotic order γ.

Fact 1. If β satisfies Assumption 1, then:

$$\lim_{P \to \infty} \mathcal{G}/\mathcal{L} = e^{-\gamma} \tag{10}$$

$$\lim_{P \to \infty} \mathcal{A}/\mathcal{L} = \frac{1}{1 + \gamma}. \tag{11}$$

Proof: We can equivalently examine the ratios in the limit as $\Omega \to \infty$, because Assumption 1 guarantees that the water-pouring bandwidth $\Omega(P)$ is a non-decreasing,

2. That is, there exists an $\Omega < \infty$ such that, for all $\omega > \Omega$, its derivative $\beta'(\omega)$ exists.

unbounded function of P. Because γ exists, β is eventually increasing, and because β is eventually differentiable, β is eventually continuous, and so $\mathcal{L} \to \beta(\Omega)$. Thus, the problem reduces to: given a function $\beta(\Omega)$ satisfying Assumption 1, what happens to the ratios of its arithmetic and geometric means over the interval $[0, \Omega]$ to its value at the endpoint $\beta(\Omega)$ as $\Omega \to \infty$?

The answer follows directly from L'Hôpital's rule. First, we prove (10). Let $g(\Omega) = \mathcal{G}(\Omega)/\beta(\Omega)$, so that:

$$\log[g(\Omega)] = \frac{1}{\Omega}\int_0^{\Omega} \log\left(\beta(\omega)/\beta(\Omega)\right) d\omega \,. \tag{12}$$

In the limit as $\Omega \to \infty$, L'Hôpital's rule yields:

$$\lim_{\Omega \to \infty} \log[g(\Omega)] = \lim_{\Omega \to \infty} \int_0^{\Omega} \frac{\partial}{\partial \Omega} \log\left(\beta(\omega)/\beta(\Omega)\right) d\omega$$

$$= \lim_{\Omega \to \infty} -\Omega\, \beta'(\Omega)/\beta(\Omega) = -\gamma \,. \tag{13}$$

This yields the desired result: $\lim_{\Omega \to \infty} g(\Omega) = e^{-\gamma}$.

We now prove (11). Let

$$h(\Omega) = \frac{\mathcal{A}(\Omega)}{\beta(\Omega)} = \frac{1}{\Omega\beta(\Omega)} \int_0^{\Omega} \beta(\omega)\, d\omega \,. \tag{14}$$

Applying L'Hôpital's rule yields the desired result:

$$\lim_{\Omega \to \infty} h(\Omega) = \lim_{\Omega \to \infty} \frac{\beta(\Omega)}{\Omega\beta'(\omega) + \beta(\Omega)} = \frac{1}{1+\gamma} \,. \tag{15}$$

For strictly bandlimited channels, the asymptotic order of (9) is undefined. Comparing the following fact with Fact 1, however, we see that a strictly bandlimited channel can be thought of as having an infinite γ.

Fact 2. For strictly bandlimited channels:

$$\lim_{P \to \infty} \mathcal{G}/\mathcal{L} = \lim_{P \to \infty} \mathcal{A}/\mathcal{L} = 0 \,. \tag{16}$$

Proof: Omitted for brevity; see [5].

3.2 Capacity at High Signal Powers

Substituting (10) into (6), we see that the asymptotic capacity (in bits/sec/Hz) in the limit as $P \to \infty$ for any channel satisfying Assumption 1 is:

$$CT \to \frac{1}{2}\log_2 e^{\gamma} = \gamma/\ln 4 \,. \tag{17}$$

Thus, the asymptotic order γ completely characterizes the behavior of capacity at large signal powers: capacity is asymptotically proportional to γ. It is interesting that only the asymptotic order of β is important, because this implies that the channel response at low frequencies is irrelevant.

4. CAPACITY PENALTY DUE TO ZF-DFE

In this section we introduce the notion of an ideal ZF-DFE channel, which epitomizes a practical zero-forcing DFE system, and compare its capacity to that of the underlying continuous-time channel.

In Fig. 1-a we illustrate how to overlay a digital communication system onto the original underlying continuous-time channel of (1) without reducing the capacity. The input signal $x(t)$ is generated from a sequence of symbols x_k using a unit-energy ideal low-pass interpolation filter with frequency response $H(\omega)$ and cut-off frequency π/T. A power constraint on $x(t)$ of P implies an energy constraint on x_k of $E[x_k^2] \le PT$. The receiver front end consists of a matched filter followed by a sampler followed by a discrete-time noise-whitening "feedforward" filter with transfer function $(1 + D(z))/F(z)$, where $F(z)$ is the analytical continuation of $F(e^{j\omega T}) = |B(\omega)|^2$ for $\omega \in [-\pi/T, \pi/T]$ and $1 + D(z)$ is the canonical (monic, causal, minimum-phase) spectral factorization of $F(z)$, satisfying:

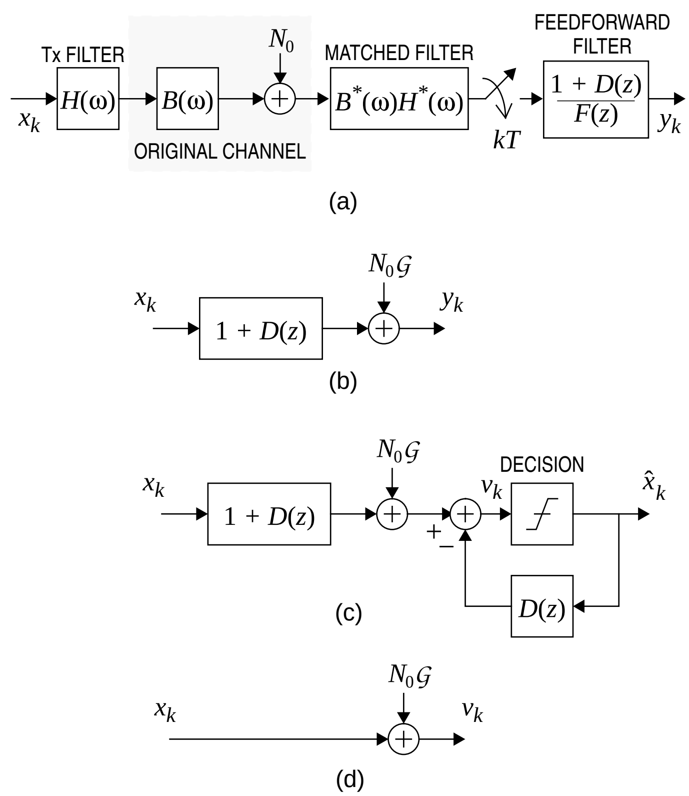

Fig. 1. Derivation of the ideal ZF-DFE channel: (a) composite channel; (b) whitened-matched filter channel; (c) structure of ZF-DFE; (d) the ideal ZF-DFE channel.

$$F(z) = \frac{1}{\mathcal{G}}(1 + D(z))\,(1 + D^*(1/z^*))\,, \qquad (18)$$

where again $\mathcal{G}$ is the geometric mean of β. We note that $F(z)$ is guaranteed to have such a factorization, because the channel $B(\omega)$ is finite and nonzero for all $\omega \in [-\pi/T, \pi/T]$; this implies that $\log F(e^{j\omega T})$ is integrable, the Paley-Wiener condition for factorizability [8].

The composite channel of Fig. 1-a reduces to the well-known whitened-matched-filter (WMF) channel of Fig. 1-b [9][10]. The equivalent discrete-time transfer function of the WMF channel is $1 + D(z)$, where $D(z)$ is strictly causal and hence represents the impulse response "tail." The additive Gaussian noise of the WMF channel is white with PSD $N_0\mathcal{G}$.

The structure of a ZF-DFE is shown in Fig. 1-c, where past decisions are fed through a feedback filter with transfer function $D(z)$ and then subtracted from the input of the decision device. When correct symbol decisions are made, so that $\hat{x}_k = x_k$, the impulse response tail $D(z)$ is completely nullified and the ideal ZF-DFE channel of Fig. 1-d results.

Of course, in any practical system, some of the decisions will be erroneous, in which case true tail-cancellation is not possible. Also, the decision device of a practical DFE requires that the symbol alphabet be finite, and a finite alphabet can only approximate the Gaussian probability distribution functions necessary to achieve capacity. For these reasons, a DFE system can only approximate the performance of an ideal ZF-DFE channel. There may very well be a significant and fundamentally unavoidable performance degradation due to these practical considerations; however, this paper is not concerned with this degradation. Instead, we will upper bound the performance of practical ZF-DFE systems by concentrating on the performance of an ideal ZF-DFE channel.

The ideal ZF-DFE channel of Fig. 1-d is free of ISI, the input symbols x_k are constrained to have power less than PT, and the noise has power $N_0\mathcal{G}$. The capacity (in b/s) of the ideal ZF-DFE channel is therefore given by:

$$C_{ZF} = \frac{1}{2T}\log_2\left[1 + \frac{PT}{N_0\mathcal{G}}\right] \qquad (19)$$

$$= \frac{1}{2T}\log_2\left[1 + \frac{\mathcal{L} - \mathcal{A}}{\mathcal{G}}\right]. \qquad (20)$$

To achieve this capacity, the PSD of the input sequence x_k must be flat over the water-pouring band $[-\pi/T, \pi/T]$.[1]

4.1 Power Penalty

The capacity of the ideal ZF-DFE channel $C_{ZF}(P)$, as given by (20), and the capacity of the original continuous-time channel $C(P)$, as given by (7), are both function of the signal power constraint P. We define the power penalty Γ as the additional power required by the ideal ZF-DFE channel — beyond that required by the underlying channel — to achieve a given capacity. We compute it by equating $C_{ZF}(\Gamma P)$ with $C(P)$ and solving for Γ, yielding:

$$\Gamma = \frac{1 - \mathcal{G}/\mathcal{L}}{1 - \mathcal{A}/\mathcal{L}}\,. \qquad (21)$$

Equation (21) can be used to find the ZF-DFE power penalty for a given power P and bandwidth Ω. We are particularly interested, however, in the behavior of Γ at high signal powers. We consider first strictly bandlimited channels, then loosely bandlimited channels.

4.1.1 Strictly Bandlimited Channels

Fact 2 tells us that, for strictly bandlimited channels, the ratios $\mathcal{A}/\mathcal{L}$ and $\mathcal{G}/\mathcal{L}$ approach zero at high signal power. From (21), therefore, the power penalty due to ideal ZF-DFE in this case also approaches zero, $\Gamma \to 0$ dB. This somewhat surprising result says that, for strictly bandlimited channels, the amount of information lost by discarding the impulse-response tail is negligible at high signal powers, and that maximum-likelihood sequence estimation, which is known to be optimal with respect to minimizing the probability of an error event, is not necessary to approach capacity; instead, a combination of coded modulation and ideal ZF-DFE is sufficient. Of course, how close a practical ZF-DFE system can get to the performance of the ideal ZF-DFE is still an open question.

4.1.2 Loosely Bandlimited Channels

Assume a loosely bandlimited channel with bowl satisfying Assumption 1. Then Fact 1 characterizes the ratios $\mathcal{A}/\mathcal{L}$ and $\mathcal{G}/\mathcal{L}$ in the limit of high signal power, and the resulting asymptotic penalty from (21) is given by:

$$\Gamma \to (1 + 1/\gamma)(1 - e^{-\gamma}). \qquad (22)$$

The power penalty is thus seen to depend only on the asymptotic order of the channel. In Fig. 2 we plot this expression versus γ. The largest penalty is $\Gamma = 1.13$ dB, occurring near $\gamma = 1.79$.

The water-pouring bandwidth, although optimal for the underlying channel, is suboptimal for the ideal ZF-DFE channel. When the bandwidth is chosen optimally, the asymptotic power penalty becomes [5]:

1. Note the sequential nature of this claim; once a symbol rate is chosen and a ZF-DFE put in place, then the optimal input spectrum is flat across the entire Nyquist band. However, the optimal symbol rate is not necessarily twice the water-pouring bandwidth; this idea is explored in [5].

$$\Gamma_{opt} \rightarrow \frac{e^{-\gamma} f^{-1}\left(e^{\gamma+1}\right)(1+1/\gamma)}{\left[\frac{1}{\gamma}\log\left(1+f^{-1}\left(e^{\gamma+1}\right)\right)\right]^{\gamma+1}}\ , \tag{23}$$

where $f: [0,\infty) \rightarrow [e,\infty)$ is defined by $f(x) = (1+x)^{(1+1/x)}$ and f^{-1} is its inverse. The lower curve in Fig. 2 was calculated using (23). In this case, the maximum penalty is $\Gamma_{opt} = 0.59$ dB, occurring at $\gamma = 1.58$.

4.2 Capacity Penalty

In this section we examine the capacity penalty due to ideal ZF-DFE, assuming the bandwidth Ω is equal to the water-pouring bandwidth. Although not optimal, we adopt this bandwidth choice because it simplifies analysis and leads to more tangible results. It can be shown that our results will not change qualitatively under bandwidth optimization [5].

Comparing (20) with (6), and recalling that $\mathcal{A} \geq \mathcal{G}$, we conclude that the capacity of the ZF-DFE channel is always less than the capacity of the original channel, with equality only when $\mathcal{A} = \mathcal{G}$; *i.e.*, when the channel is flat and there is no ISI. The capacity reduction due to ZF-DFE can be written as:

$$\Delta C \equiv C - C_{ZF} = -\frac{1}{2T}\log_2\left[1+\frac{\mathcal{G}}{\mathcal{L}}-\frac{\mathcal{A}}{\mathcal{L}}\right]\ . \tag{24}$$

We are interested in what happens to ΔC in the limit of large signal power. Once again, the problem reduces to one of characterizing the ratios $\mathcal{G}/\mathcal{L}$ and $\mathcal{A}/\mathcal{L}$. We consider first strictly, then loosely bandlimited channels.

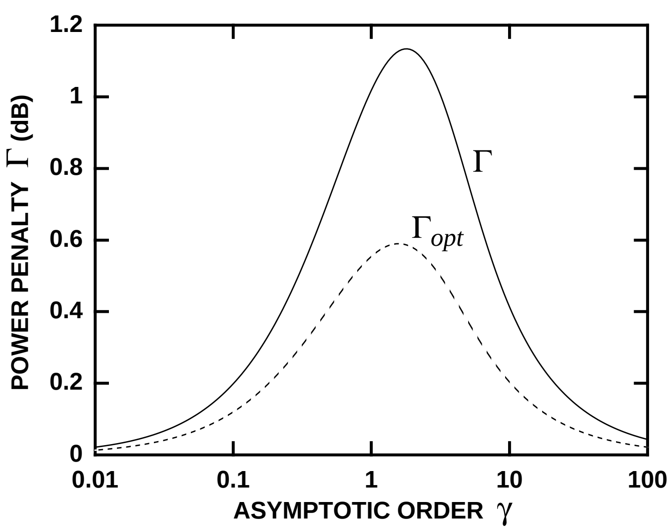

Fig. 2. Asymptotic power penalties with (dashed) and without (solid) bandwidth optimization.

4.2.1 Strictly Bandlimited Channels

Assume a strictly bandlimited channel with cutoff ω_{max}. Because of the bandwidth restriction, the coefficient $1/(2T)$ in (24) can never be greater than $\omega_{max}/(2\pi)$. From Fact 2 we know that the ratios $\mathcal{A}/\mathcal{L}$ and $\mathcal{G}/\mathcal{L}$ approach zero in the limit of large signal power. Thus, the argument of the logarithm in (24) approaches unity, and therefore $\Delta C \rightarrow 0$ as $P \rightarrow \infty$.

4.2.2 Loosely Bandlimited Channels

We now show that, if β satisfies Assumption 1 with $\gamma \in (0, \infty)$, then $\Delta C \rightarrow \infty$ as $P \rightarrow \infty$.

Assume that β satisfies Assumption 1. Using Fact 1 in (6) and (20), the capacities of the original and ideal ZF-DFE channels in the limit of large signal power become:

$$C \rightarrow \frac{1}{2T}\log_2(e^{\gamma}) \tag{25}$$

$$C_{ZF} \rightarrow \frac{1}{2T}\log_2\left(\frac{\gamma}{\gamma+1}e^{\gamma}+1\right) \tag{26}$$

and the difference:

$$\Delta C \rightarrow -\frac{1}{2T}\log_2\left(\frac{\gamma}{\gamma+1}+e^{-\gamma}\right). \tag{27}$$

From (27) we see that, when $\gamma \in (0,\infty)$, ΔC does not go to zero, as it did for strictly bandlimited channels. In fact, because $1/T \rightarrow \infty$ as $P \rightarrow \infty$, the capacity penalty grows without bound ($\Delta C \rightarrow \infty$)! Of course, both C and C_{ZF} are growing without bound as well, so an unbounded ΔC is not necessarily severe. In this case, a different measure of the capacity penalty is useful: the ratio C_{ZF}/C. An expression for the ratio C_{ZF}/C in the limit of large signal power can be found by dividing (26) by (25). This asymptotic ratio is plotted in Fig. 3. The largest penalty is about 14.2%, occurring near $\gamma = 0.96$; the penalty is not as large for channels that roll off either faster or slower.

With bandwidth optimization, the asymptotic capacity $C_{ZF,opt}$ of the ideal ZF-DFE channel will be slightly larger than C_{ZF} as given above, and can be shown to be [5]:

$$C_{ZF,opt} = \frac{\frac{1}{2T}\log_2\left(1+f^{-1}\left(e^{\gamma+1}\right)\right)}{\left[e^{-\gamma}f^{-1}\left(e^{\gamma+1}\right)(1+1/\gamma)\right]^{\frac{1}{1+\gamma}}}\ . \tag{28}$$

The upper curve in Fig. 3 illustrates the improvements possible through bandwidth optimization.

The results of Fig. 2 and Fig. 3 are sufficient to characterize the asymptotic capacity penalties for a broad range of channels, including all channels with proper, rational transfer functions. For example, a second-order Butterworth filter has an asymptotic order of $\gamma = 4$, and hence an ideal ZF-DFE system suffers an asymptotic power penalty of 0.44 dB (0.89 dB) and an asymptotic capacity penalty of 2% (5%), with (without) bandwidth optimization.

5. SUMMARY AND CONCLUSIONS

We showed how the water-pouring procedure for calculating channel capacity is related to the arithmetic and geometric means of the inverse folded spectrum. We defined the asymptotic order γ of a channel and showed that, when γ exists, the optimal spectral efficiency (in b/symbol or b/s/Hz) approaches $\gamma / \ln 4$ at large signal powers. We defined an ideal ZF-DFE channel and compared its capacity C_{ZF} to the capacity of the underlying channel.

The capacity of the ideal ZF-DFE channel, despite the assumptions of infinite-length filters, zero-delay correct decisions, and arbitrary symbol alphabet, was strictly less than the capacity of the underlying channel, which leads to the conclusion that ZF-DFE is a suboptimal equalization technique. On loosely bandlimited channels, the capacity penalty is non-zero even in the limit of infinite signal power.

The tightness of C_{ZF} as an upper bound to reliable bit rates using a practical ZF-DFE system is still an open problem; a complete analysis should take into account the effects of error propagation, decoding delay, and finite symbol alphabet, all of which separate the ideal ZF-DFE from a practical one. In addition, the capacity penalty, if any, due to minimum-mean-squared-error DFE has not yet been characterized.

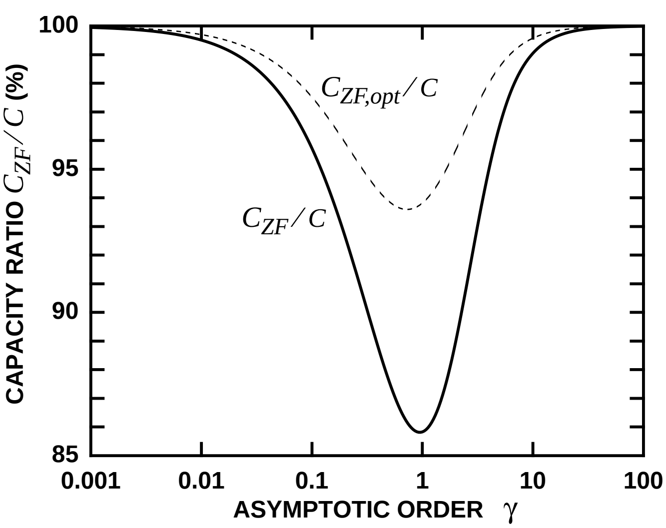

Fig. 3. Asymptotic capacity of ideal ZF-DFE channel as a percentage of the capacity of the underlying channel: with (dashed) and without (solid) bandwidth optimization.

6. ACKNOWLEDGMENTS

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