EE6604
Personal & Mobile Communications

Week 11

Continuous Phase Modulation

Reading: 4.7, 4.8

Power Spectrum of Digitally Modulated Signals

Reading: 4.9

Continuous Phase Modulation (CPM)

- The CPM bandpass signal is

$$\begin{aligned} s(t) &= \mathrm{Re}\left\{Ae^{j\phi(t)}e^{j2\pi f_c t}\right\} \\ &= A\cos\left(2\pi f_c t + \phi(t)\right) \end{aligned} \quad (1)$$

  where the "excess phase" is

$$\phi(t) = 2\pi h \int_0^t \sum_{k=0}^{\infty} x_k h_f(\tau - kT)d\tau$$

  - h is the modulation index
  - $x_n \in \{\pm 1, \pm 3, \ldots, \pm(M-1)\}$ are the M-ary data symbols
  - $h_f(t)$ is the "**frequency shaping pulse**" of duration LT, that is zero for $t < 0$ and $t > LT$, and normalized to have an area equal to 1/2. Full response CPM has $L = 1$, while partial response CPM has $L > 1$.

- The instantaneous frequency deviation from the carrier is

$$f_{\mathrm{dev}}(t) = \frac{1}{2\pi}\frac{\mathrm{d}\phi(t)}{\mathrm{d}t} = h\sum_{k=0}^{\infty} x_k h_f(t - kT) \ .$$

Frequency Shaping Pulses

pulse type	$h_f(t)$
L-rectangular (LREC)	$\frac{1}{2LT} u_{LT}(t)$
L-raised cosine (LRC)	$\frac{1}{2LT}\left[1 - \cos\left(\frac{2\pi t}{LT}\right)\right] u_{LT}(t)$
L-half sinusoid (LHS)	$\frac{\pi}{4LT}\sin(\pi t/LT) u_{LT}(t)$
L-triangular (LTR)	$\frac{1}{LT}\left(1 - \frac{\lvert t-LT/2 \rvert}{LT/2}\right)$

Excess Phase and Tilted Phase

- During the time interval $nT \leq t \leq (n+1)T$, the **excess phase** $\phi(t)$ is

$$\phi(t) = 2\pi h \sum_{k=0}^{n} x_k \beta(t - kT).$$

where the "**phase shaping pulse**" is

$$\beta(t) = \begin{cases} 0 & , \ t < 0 \\ \int_0^t h_f(\tau) d\tau & , \ 0 \leq t \leq LT \\ 1/2 & , \ t \geq LT \end{cases}$$

- For the case of full response CPM $(L = 1)$, during the time interval $nT \leq t \leq (n+1)T$ the excess phase is

$$\phi(t) = \pi h \sum_{k=0}^{n-1} x_k + 2\pi h x_n \beta(t - nT)$$

- During the time interval $nT \leq t \leq (n+1)T$, the CPM "**tilted phase**" is

$$\begin{aligned} \psi(t) &= \pi h \sum_{k=0}^{n-1} x_k + 2\pi h x_n \beta(t - nT) + \pi h(M-1)t/T \\ &= \phi(t) + \pi h(M-1)t/T \end{aligned}$$

- Note that $s(t)$ can be generated by replacing $\phi(t)$ with $\psi(t)$ and f_c by $f_c - h(M-1)/2T$ in (1).

Continuous Phase Frequency Shift Keying (CPFSK)

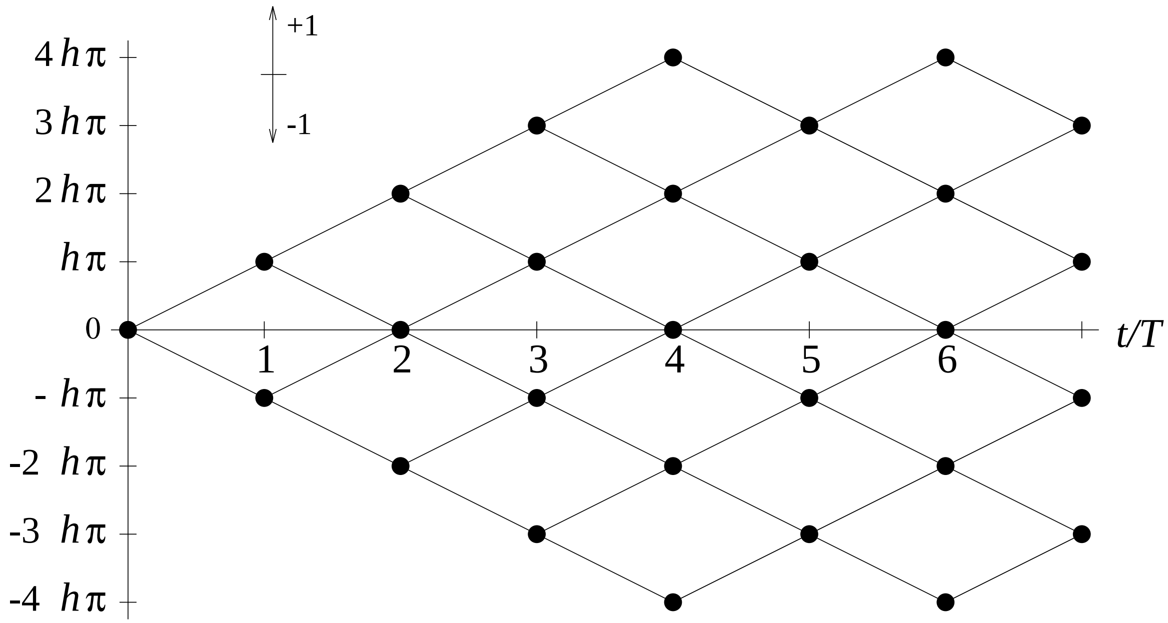
- Continuous phase frequency shift keying (CPFSK) is a special type of CPM that uses the full response REC shaping function

$$h_f(t) = \frac{1}{2T} u_T(t) = \frac{1}{2T}\left(u(t) - u(t-T)\right)$$

As a result

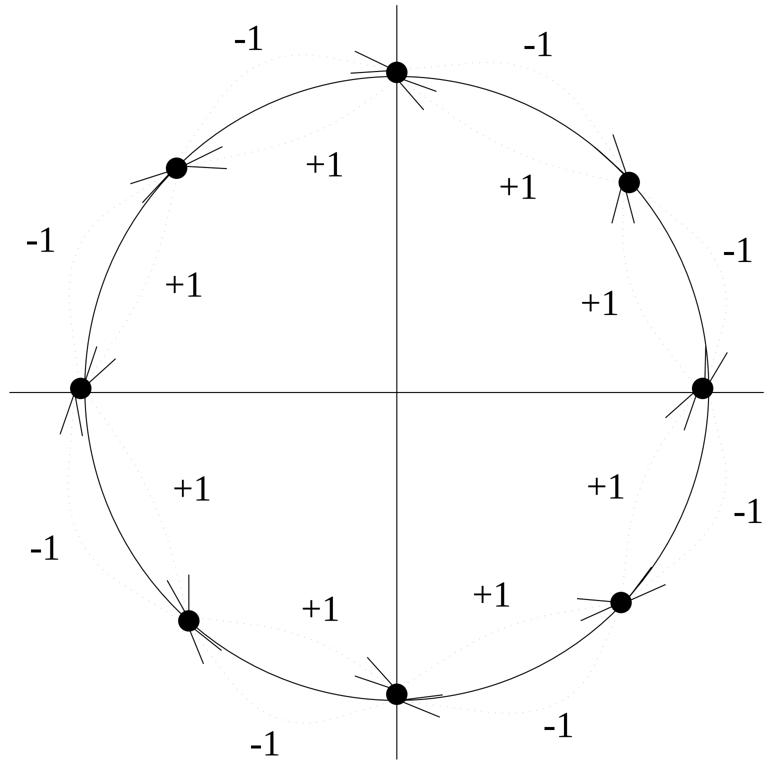
$$\beta(t) = \begin{cases} 0 & , \; t < 0 \\ t/2T & , \; 0 \leq t \leq T \\ 1/2 & , \; t \geq T \end{cases}$$

- Since the frequency shaping function is rectangular, the phase shaping pulse contains a linear ramp and the CPFSK excess phase trajectories are linear.



Phase tree of binary CPFSK.

Phase-state Diagrams



Phase-state diagram of CPM with $h = 1/4$.

Minimum Shift Keying (MSK)

- MSK is a special case of CPFSK, where the modulation index $h = \frac{1}{2}$ is used.
- The phase shaping pulse is

$$\beta(t) = \begin{cases} 0 & , \; t < 0 \\ t/2T & , \; 0 \le t \le T \\ 1/2 & , \; t \ge T \end{cases}$$

- The MSK bandpass waveform is

$$s(t) = A \cos\left(2\pi f_c t + + \frac{\pi}{2} \sum_{k=0}^{n-1} x_k + \frac{t - nT}{2T} \pi x_n\right) \quad , \quad nT \le t \le (n+1)T$$

- The **excess phase** on the interval $nT \le t \le (n+1)T$ is

$$\phi(t) = \frac{\pi}{2} \sum_{k=0}^{n-1} x_k + \frac{t - nT}{2T} \pi x_n$$

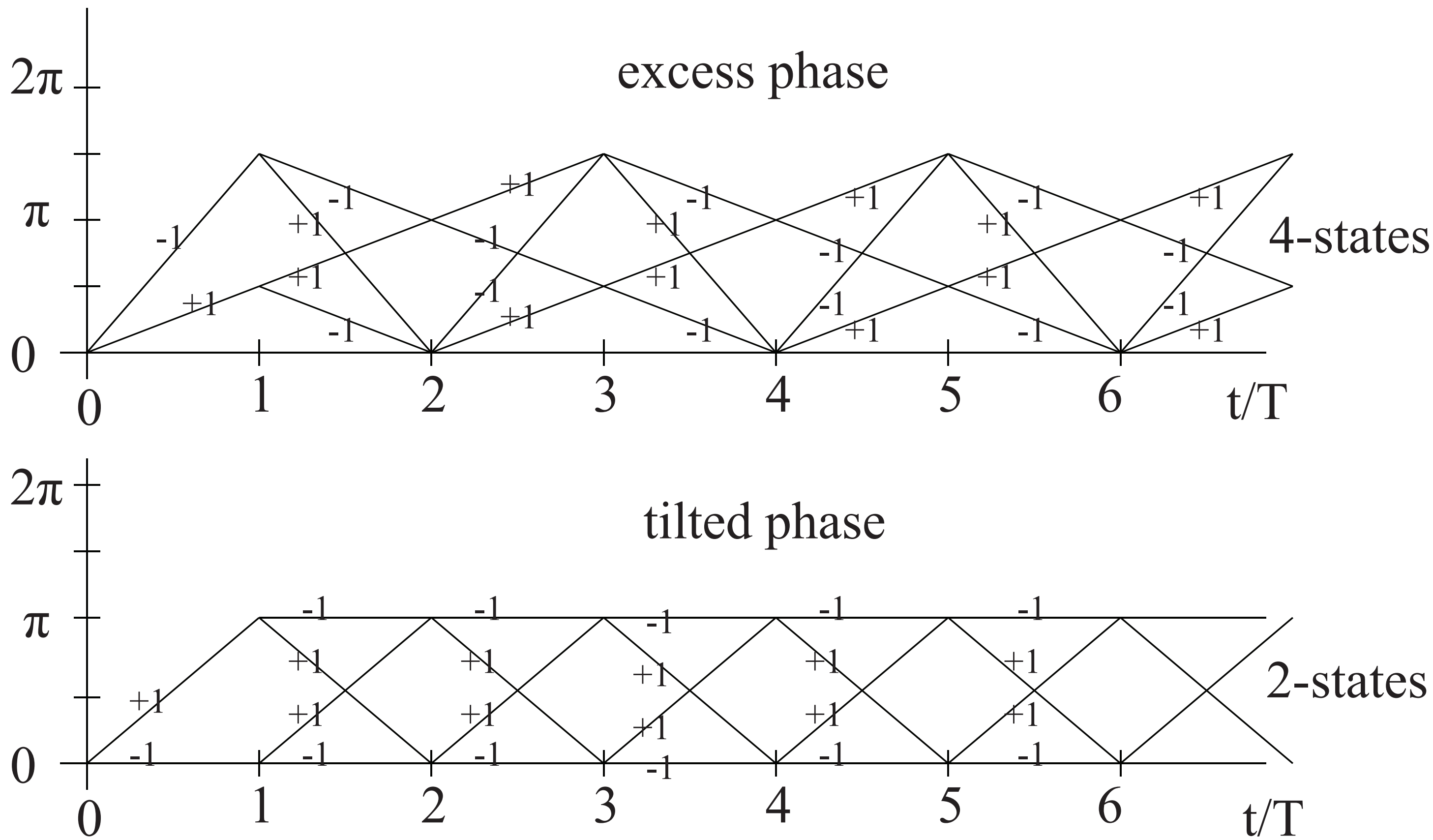
- The **tilted phase** on the interval $nT \le t \le (n+1)T$ is

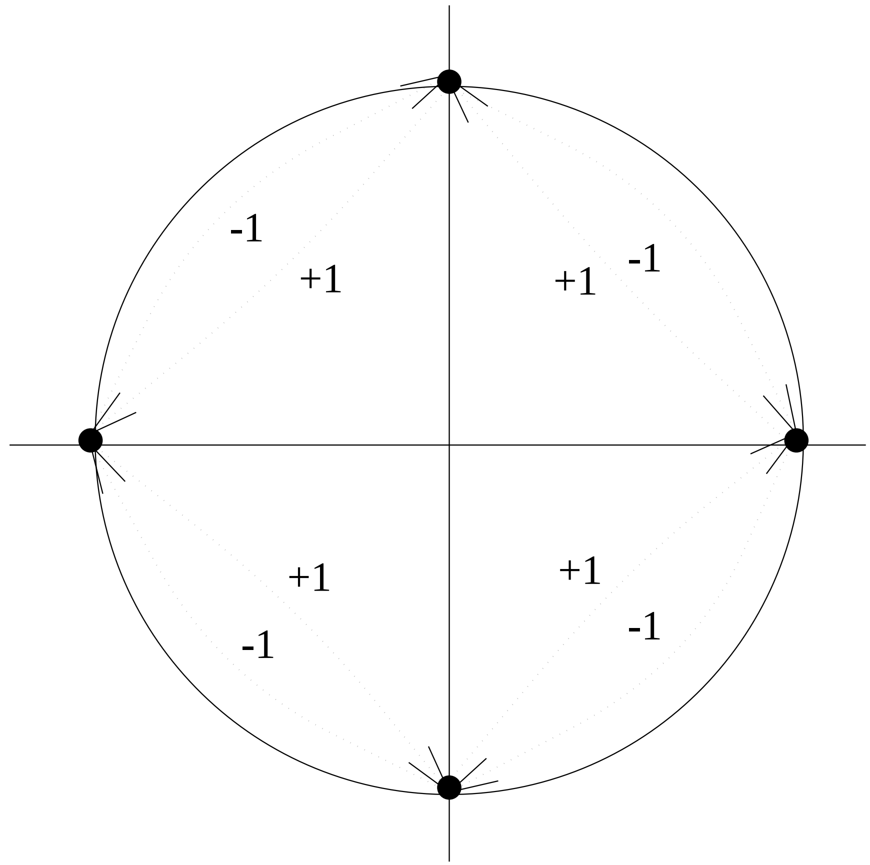
$$\psi(t) = \phi(t) + \frac{\pi t}{2T}$$

- Combining the above two equations, we have

$$\psi((n+1)T) = \psi(nT) + \frac{\pi}{2}(1 + x_n)$$

Excess Phase and Tilted Phase for Minimum Shift Keying (MSK)





Phase state diagram for MSK signals.

Linearized Representation of MSK

- An interesting representation for MSK waveforms can be obtained by using Laurent's decomposition to express the MSK complex envelope in the quadrature form

$$\tilde{s}(t) = A \sum_n b(t - 2nT, \mathbf{x}_n) \ ,$$

where

$$b(t, \mathbf{x}_n) = \hat{x}_{2n+1} h_a(t - T) + j \hat{x}_{2n} h_a(t)$$

and where $\mathbf{x}_n = (\hat{x}_{2n+1}, \hat{x}_{2n})$,

$$\hat{x}_{2n} = \hat{x}_{2n-1} x_{2n} \qquad (2)$$

$$\hat{x}_{2n+1} = -\hat{x}_{2n} x_{2n+1} \qquad (3)$$

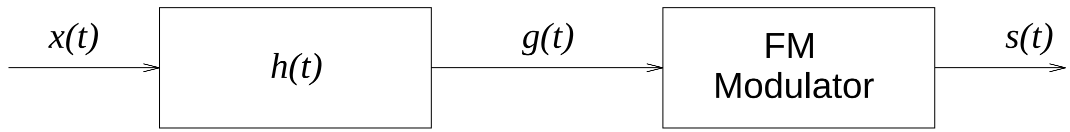
$$\hat{x}_{-1} = 1 \qquad (4)$$

and

$$h_a(t) = \sin\left(\frac{\pi t}{2T}\right) u_{2T}(t) \ .$$

- The sequences, $\{\hat{x}_{2n}\}$ and $\{\hat{x}_{2n+1}\}$, are independent binary symbol sequences taking on elements from the set $\{-1, +1\}$.
- The symbols $\hat{x}_{2n}$ and $\hat{x}_{2n+1}$ are transmitted on the quadrature branches with a half-sinusoid (HS) amplitude shaping pulse of duration $2T$ seconds and an offset of T seconds.

Gaussian MSK (GMSK)



Gaussian Pre-modulation filtered MSK (GMSK).

- With MSK the modulating signal is

$$x(t) = \frac{1}{2T} \sum_{n=-\infty}^{\infty} x_n u_T(t - nT)$$

- The bandwidth of $\tilde{s}(t)$ depends on the bandwidth of $x(t)$ and the modulation index h. For GMSK $h = 1/2$.
- We filter $x(t)$ with a low-pass filter to remove high frequency content prior to modulation, i.e., we use the filtered pulse $g(t) = x(t) * h(t)$.
- For GMSK, the low-pass filter transfer function is

$$H(f) = \exp\left\{ -\left(\frac{f}{B}\right)^2 \frac{\ln 2}{2} \right\}$$

where B is the 3 dB filter bandwidth.

Gaussian Pre-modulation filtered MSK (GMSK).

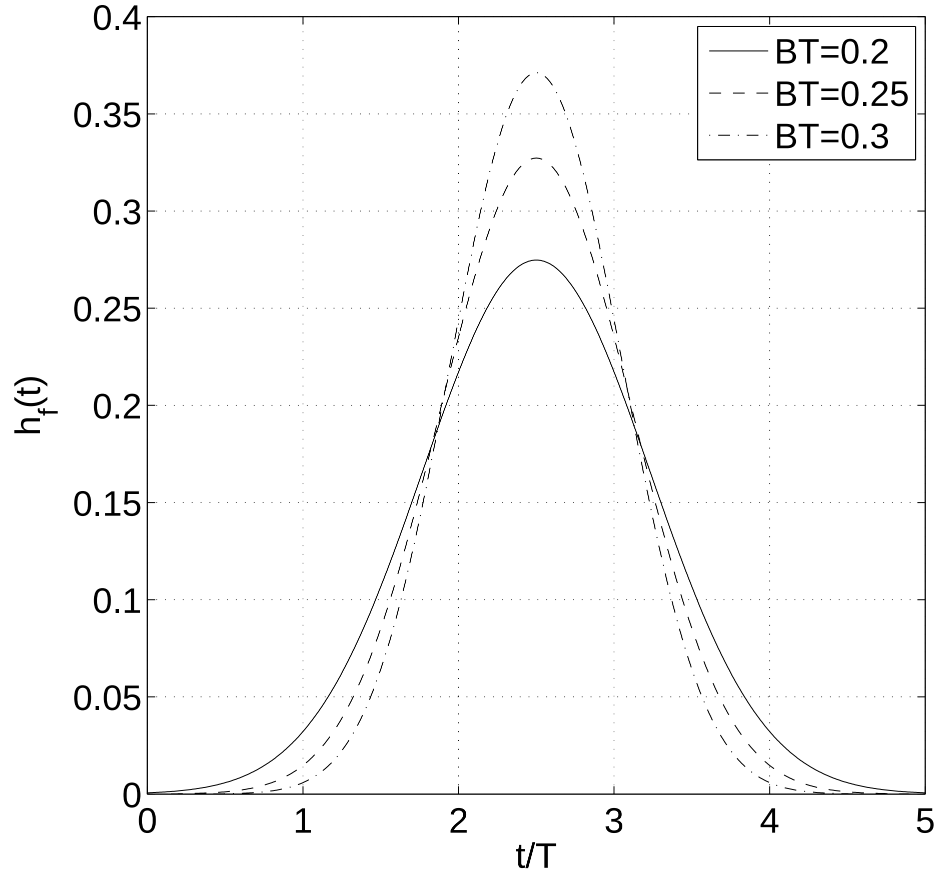
- A rectangular pulse $\mathrm{rect}(t/T) = u_T(t + T/2)$ transmitted through this Gaussian low-pass filter yields the GMSK frequency shaping pulse

$$
\begin{aligned}
h_f(t) &= \frac{1}{2T}\sqrt{\frac{2\pi}{\ln 2}}(BT)\int_{t/T-1/2}^{t/T+1/2} \exp\left\{-\frac{2\pi^2(BT)^2x^2}{\ln 2}\right\} dx \\
&= \frac{1}{2T}\left[Q\left(\frac{t/T-1/2}{\sigma}\right) - Q\left(\frac{t/T+1/2}{\sigma}\right)\right]
\end{aligned}
$$

where

$$
\begin{aligned}
Q(\alpha) &= \int_\alpha^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \\
\sigma^2 &= \frac{\ln 2}{4\pi^2(BT)^2} \ .
\end{aligned}
$$

- The total pulse area is $\int_{-\infty}^{\infty} h_f(t)dt = 1/2$ and, therefore, the total contribution to the excess phase for each data symbol is $\pm\pi/2$ radians.



GMSK frequency shaping pulse for various normalized filter bandwidths BT.

- The GMSK phase shaping pulse is

$$\beta(t) = \int_{-\infty}^{t} h_f(t)dt = \frac{1}{2}\left(G\left(\frac{t}{T}+\frac{1}{2}\right) - G\left(\frac{t}{T}-\frac{1}{2}\right)\right)$$

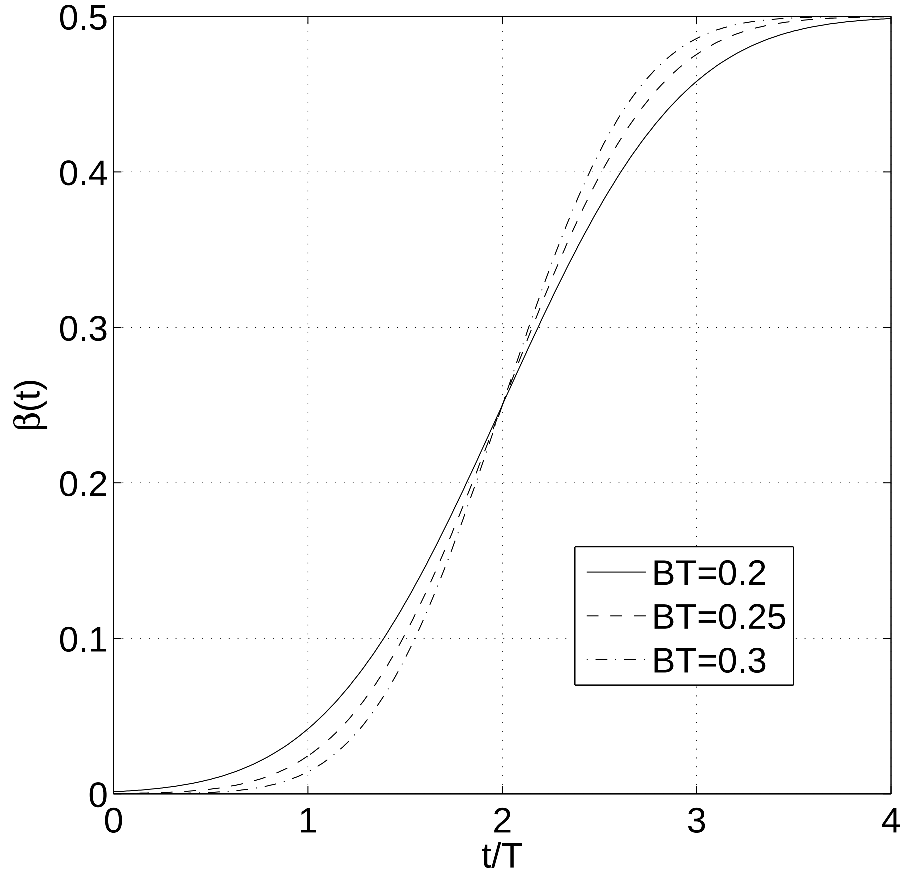
where

$$G(x) = x\ \Phi\left(\frac{x}{\sigma}\right) + \frac{\sigma}{\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}\ ,$$

and

$$\Phi(\alpha) = \int_{-\infty}^{\alpha} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

- Observe that $\beta(\infty) = 1/2$ and, therefore, the total contribution to the excess phase for each data symbol remains at $\pm\pi/2$ as mentioned earlier.



GMSK phase shaping pulse for various normalized filter bandwidths BT.

- The excess phase change over the interval from $-T/2$ to $T/2$ is

$$\phi(T/2) - \phi(-T/2) = x_0\beta_0(T) + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} x_n\beta_n(T)$$

  where

$$\beta_n(T) = \int_{-T/2-nT}^{T/2-nT} h_f(\nu)\, d\nu \ .$$

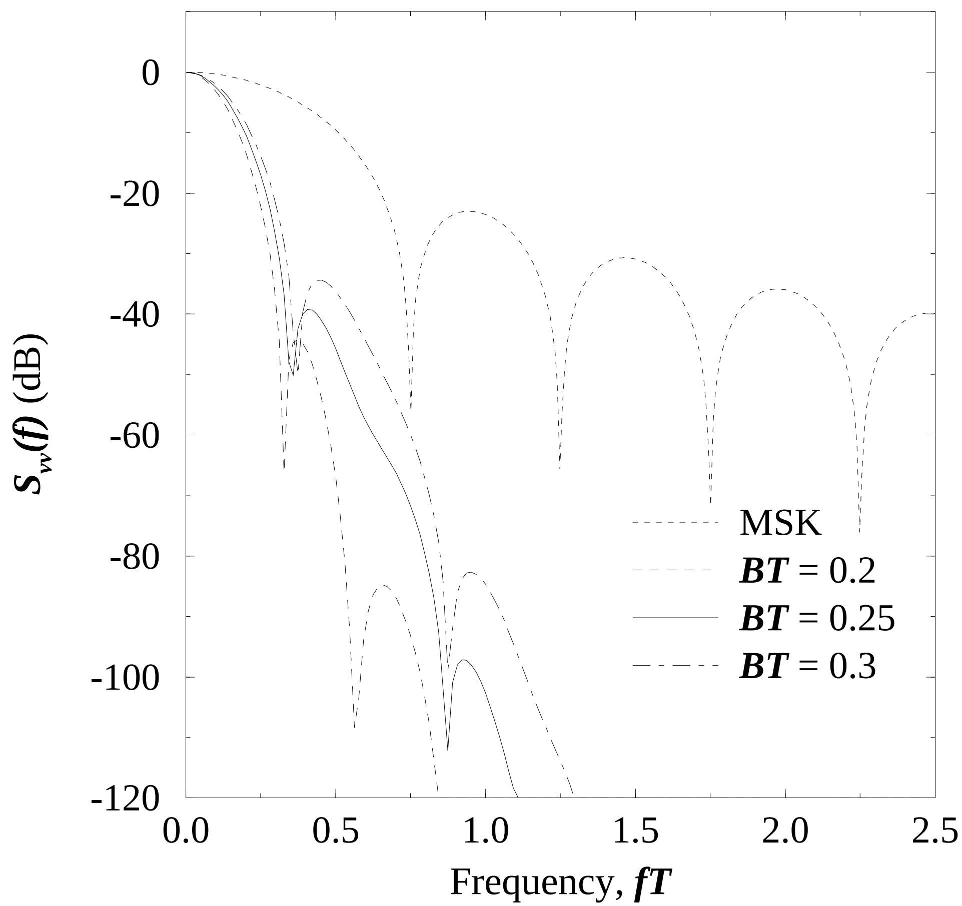
  and

$$h_f(t) = \frac{1}{2T}\left[Q\left(\frac{t/T - 1/2}{\sigma}\right) - Q\left(\frac{t/T + 1/2}{\sigma}\right)\right]$$

- The first term, $x_0\beta_0(T)$ is the desired term, and the second term, $\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} x_n\beta_n(T)$, is the intersymbol interference (ISI) introduced by the Gaussian low-pass filter.

- **Conclusion:** GMSK trades off power efficiency (due to the induced ISI) for a greatly improved bandwdith efficiency.

  – the loss in power efficiency can be recovered by using an equalizer in the receiver to mitigate the induced ISI.



Power spectral density of GMSK with various normalized filter bandwidths BT.

Linearized Gaussian Minimum Shift Keying (LGMSK)

- Laurent showed that any binary partial response CPM signal can be represented exactly as a linear combination of 2^{L-1} partial-response pulse amplitude modulated (PAM) signals, viz.,

$$\tilde{s}(t) = \sum_{n=0}^{\infty} \sum_{p=0}^{2^{L-1}-1} e^{j\pi h \alpha_{n,p}} c_p(t - nT),$$

where

$$\begin{aligned} c_p(t) &= c(t) \prod_{n=1}^{L-1} c\left(t + (n + L\varepsilon_{n,p})T\right), \\ \alpha_{n,p} &= \sum_{m=0}^{n} x_m - \sum_{m=1}^{L-1} x_{n-m}\varepsilon_{m,p}, \end{aligned}$$

and $\varepsilon_{n,p} \in \{0, 1\}$ are the coefficients of the binary representation of the index p, i.e.,

$$p = \varepsilon_{0,p} + 2\varepsilon_{1,p} + \cdots + 2^{L-2}\varepsilon_{L-2,p} \ .$$

- The basic signal pulse $c(t)$ is

$$c(t) = \begin{cases} \frac{\sin(2\pi h \beta(t))}{\sin \pi h} & , \quad 0 \le t < LT \\ \frac{\sin(\pi h - 2\pi h \beta(t - LT))}{\sin \pi h} & , \quad LT \le t < 2LT \\ 0 & , \quad \text{otherwise} \end{cases} ,$$

where $\beta(t)$ is the CPM phase shaping function.

Linearized Gaussian Minimum Shift Keying (LGMSK)

- Note that the GMSK frequency shaping pulse spans $L = 3$ to $L = 4$ symbol periods for practical values of BT.
- Often the pulse $c_0(t)$ contains most of the signal energy, so the $p = 0$ term in can provide a good approximation to the CPM signal. Numerical analysis can show that the pulse $c_0(t)$ contains 99.83% of the energy and, therefore, we can derive a linearized GMSK waveform by using only $c_0(t)$ and neglecting the other pulses.
- This yields the waveform

  $$\tilde{s}(t) = \sum_{n=0}^{\infty} e^{j\pi h \alpha_{n,o}} c_0(t - nT),$$

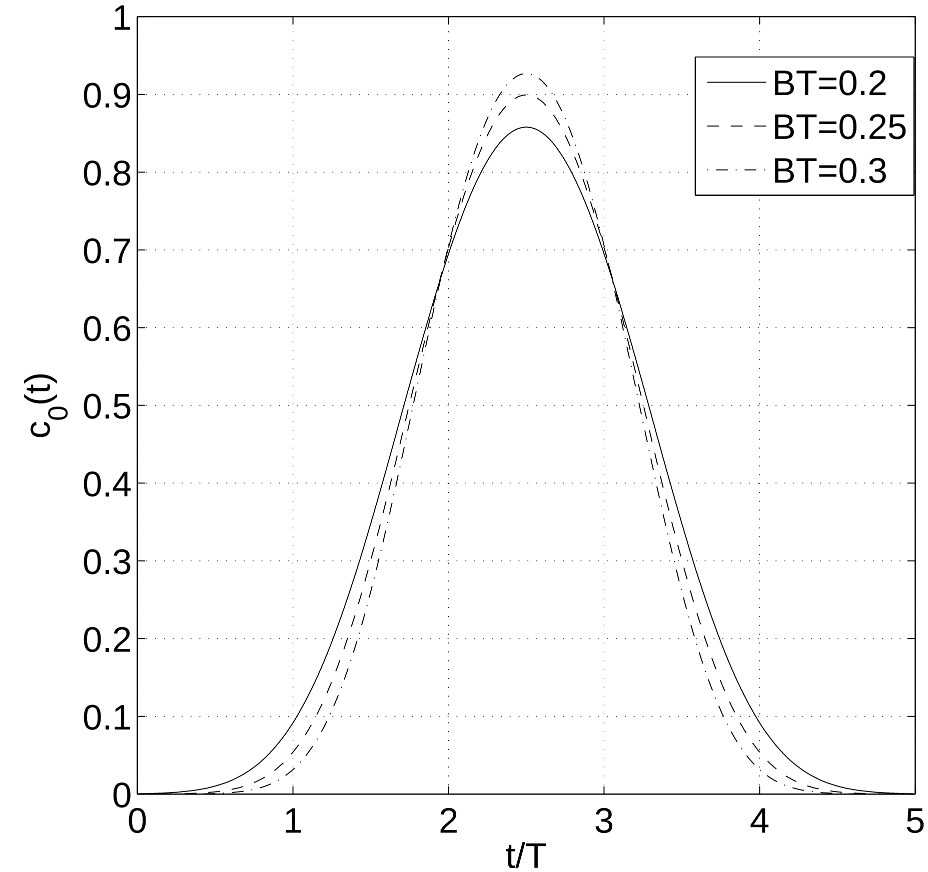
  where, with $L = 4$,

  $$\begin{aligned} c_0(t) &= \prod_{n=0}^{3} c\left(t + nT\right), \\ \alpha_{n,0} &= \sum_{m=0}^{n} x_m \end{aligned}$$

- Since the GMSK phase shaping pulse is non-causal, when evaluating $c(t)$ we use the truncated and time shifted GMSK phase shaping pulse

  $$\hat{\beta}(t) = \beta(t - 2T)$$

  with $L = 4$ as shown previously.



LGMSK amplitude shaping pulse for various normalized premodulation filter bandwidths BT.

Linearized Gaussian Minimum Shift Keying (LGMSK)

- For $h = 1/2$ used in GMSK,

$$a_{n,0} = e^{j\frac{\pi}{2}\alpha_{n,0}} \in \{\pm 1, \pm j\} \ ,$$

  and it follows that

$$\tilde{s}(t) = A \sum_n \Big(\hat{x}_{2n+1} c_0(t - 2nT - T) + j \hat{x}_{2n} c_0(t - 2nT) \Big)$$

  where

$$\begin{aligned} \hat{x}_{2n} &= \hat{x}_{2n-1} x_{2n} \\ \hat{x}_{2n+1} &= -\hat{x}_{2n} x_{2n+1} \\ \hat{x}_{-1} &= 1 \end{aligned}$$

- This is the same as the OQPSK representation for MSK except that the half-sinusoid amplitude pulse shaping function is replaced with the LGMSK amplitude pulse shaping function.
- Note that the LGMSK pulse has length of approximately $3T$ to $4T$, while the pulses on the quadrature branches are transmitted every $2T$ seconds. Therefore, the LGMSK pulse introduces ISI, but this can be corrected with an equalizer in the receiver.

POWER SPECTRUM OF BANDPASS SIGNALS

- A bandpass modulated signal can be written in the form

$$
\begin{aligned}
s(t) &= \Re\left\{\tilde{s}(t)e^{j(2\pi f_c t)}\right\} \\
&= \frac{1}{2}\left\{\tilde{s}(t)e^{j(2\pi f_c t)} + \tilde{s}^*(t)e^{-j(2\pi f_c t)}\right\}
\end{aligned}
$$

- The autocorrelation of a bandpass modulated signal is

$$
\begin{aligned}
\phi_{ss}(\tau) &= \mathrm{E}\left[s(t)s(t+\tau)\right] \\
&= \frac{1}{4}\mathrm{E}\left[\left(\tilde{s}(t)e^{j2\pi f_c t} + \tilde{s}^*(t)e^{-j2\pi f_c t}\right)\right. \\
&\quad \left.\times\left(\tilde{s}(t+\tau)e^{j(2\pi f_c t+2\pi f_c\tau)} + \tilde{s}^*(t+\tau)e^{-j(2\pi f_c t+2\pi f_c\tau)}\right)\right] \\
&= \frac{1}{4}\mathrm{E}\left[\tilde{s}(t)\tilde{s}(t+\tau)e^{j(4\pi f_c t+2\pi f_c\tau)} + \tilde{s}(t)\tilde{s}^*(t+\tau)e^{-j2\pi f_c\tau}\right. \\
&\quad \left. + \tilde{s}^*(t)\tilde{s}(t+\tau)e^{j2\pi f_c\tau} + \tilde{s}^*(t)\tilde{s}^*(t+\tau)e^{-j(4\pi f_c t+2\pi f_c\tau)}\right] \\
&= \frac{1}{4}\left[\mathrm{E}[\tilde{s}(t)\tilde{s}(t+\tau)]e^{j(4\pi f_c t+2\pi f_c\tau)} + \mathrm{E}[\tilde{s}(t)\tilde{s}^*(t+\tau)]e^{-j2\pi f_c\tau}\right. \\
&\quad \left.+\mathrm{E}[\tilde{s}^*(t)\tilde{s}(t+\tau)]e^{j2\pi f_c\tau} + \mathrm{E}[\tilde{s}^*(t)\tilde{s}^*(t+\tau)]e^{-j(4\pi f_c t+2\pi f_c\tau)}\right] \quad .
\end{aligned}
$$

- If $s(t)$ is a wide-sense stationary random process, then the exponential terms that involve t must vanish, i.e., $\mathrm{E}[\tilde{s}(t)\tilde{s}(t+\tau)] = 0$ and $\mathrm{E}[\tilde{s}^*(t)\tilde{s}^*(t+\tau)] = 0$.
- Substituting $\tilde{s}(t) = \tilde{s}_I(t) + j\tilde{s}_Q(t)$ into the above expectations gives the result

$$
\begin{aligned}
\phi_{\tilde{s}_I ß_I}(\tau) &= \mathrm{E}[\tilde{s}_I(t)\tilde{s}_I(t+\tau)] = \mathrm{E}[\tilde{s}_Q(t)\tilde{s}_Q(t+\tau)] = \phi_{\tilde{s}_Q\tilde{s}_Q}(\tau) \\
\phi_{\tilde{s}_I\tilde{s}_Q}(\tau) &= \mathrm{E}[\tilde{s}_I(t)\tilde{s}_Q(t+\tau)] = -\mathrm{E}[\tilde{s}_Q(t)\tilde{s}_I(t+\tau)] = -\phi_{\tilde{s}_Q\tilde{s}_I}(\tau)
\end{aligned}
$$

- Using these results, the autocorrelation is

$$
\phi_{ss}(\tau) = \frac{1}{2}\phi_{\tilde{s}\tilde{s}}(\tau)e^{j2\pi f_c\tau} + \frac{1}{2}\phi^*_{\tilde{s}\tilde{s}}(\tau)e^{-j2\pi f_c\tau}
$$

  where

$$
\phi_{\tilde{s}\tilde{s}}(\tau) = \frac{1}{2}\mathrm{E}[\tilde{s}^*(t)\tilde{s}(t+\tau)]
$$

- The power density spectrum is the Fourier transform of $\phi_{ss}(\tau)$:

$$
S_{ss}(f) = \frac{1}{2}\left[S_{\tilde{s}\tilde{s}}(f-f_c) + S^*_{\tilde{s}\tilde{s}}(-f-f_c)\right]
$$

  - $S_{\tilde{s}\tilde{s}}(f)$ is the power density spectrum of the complex envelope $\tilde{s}(t)$, which is always real-valued but not necessarily even about $f = 0$.

$$
S_{ss}(f) = \frac{1}{2}\left[S_{\tilde{s}\tilde{s}}(f-f_c) + S_{\tilde{s}\tilde{s}}(-f-f_c)\right]
$$

POWER SPECTRAL DENSITY OF A COMPLEX ENVELOPE

- In general, the complex lowpass signal is of the form

$$\tilde{s}(t) = A \sum_k b(t - kT, \mathbf{x}_k)$$

- The autocorrelation of $\tilde{s}(t)$ is

$$\begin{aligned} \phi_{\tilde{s}\tilde{s}}(t, t+\tau) &= \frac{1}{2}\mathrm{E}\left[\tilde{s}^*(t)\tilde{s}(t+\tau)\right] \\ &= \frac{A^2}{2} \sum_i \sum_k \mathrm{E}\left[b^*(t - iT, \mathbf{x}_i) b(t + \tau - kT, \mathbf{x}_k)\right] \quad . \end{aligned}$$

Observe that $\tilde{s}(t)$ is a cyclostationary random process, meaning that the autocorrelation function $\phi_{\tilde{s}\tilde{s}}(t, t+\tau)$ is periodic in t with period T. To see this property, first note that

$$\begin{aligned} &\phi_{\tilde{s}\tilde{s}}(t+T, t+T+\tau) \\ &= \frac{A^2}{2} \sum_i \sum_k \mathrm{E}\left[b^*(t + T - iT, \mathbf{x}_i) b(t + T + \tau - kT, \mathbf{x}_k)\right] \\ &= \frac{A^2}{2} \sum_{i'} \sum_{k'} \mathrm{E}\left[b^*(t - i'T, \mathbf{x}_{i'+1}) b(t + \tau - k'T, \mathbf{x}_{k'+1})\right] \quad . \end{aligned}$$

- Under the assumption that the information sequence is a stationary random process it follows that

$$\begin{aligned}\phi_{\tilde{s}\tilde{s}}(t+T, t+T+\tau) &= \frac{A^2}{2}\sum_{i'}\sum_{k'} \mathrm{E}\left[b^*(t-i'T, \mathbf{x}_{i'})b(t+\tau-k'T, \mathbf{x}_{k'})\right] \\ &= \phi_{\tilde{s}\tilde{s}}(t, t+\tau) \ . \end{aligned} \quad (5)$$

where data blocks $\mathbf{x}_{i'+1}$ and $\mathbf{x}_{k'+1}$ are replaced by $\mathbf{x}_{i'}$ and $\mathbf{x}_{k'}$, respectively. Therefore $\tilde{s}(t)$ is cyclostationary.

- Since $\tilde{s}(t)$ is cyclostationary, the autocorrelation $\phi_{\tilde{s}\tilde{s}}(\tau)$ can be obtained by taking the time average of $\phi_{\tilde{s}\tilde{s}}(t, t+\tau)$, given by

$$
\begin{aligned}
\phi_{\tilde{s}\tilde{s}}(\tau) &= < \phi_{\tilde{s}\tilde{s}}(t, t+\tau) > \\
&= \frac{A^2}{2} \sum_i \sum_k \frac{1}{T} \int_0^T \mathrm{E}\left[b^*(t - iT, \mathbf{x}_i) b(t + \tau - kT, \mathbf{x}_k)\right] dt \\
&= \frac{A^2}{2T} \sum_i \sum_k \int_{-iT}^{-iT+T} \mathrm{E}\left[b^*(z, \mathbf{x}_i) b(z + \tau - (k-i)T, \mathbf{x}_k)\right] dz \\
&= \frac{A^2}{2T} \sum_i \sum_m \int_{-iT}^{-iT+T} \mathrm{E}\left[b^*(z, \mathbf{x}_i) b(z + \tau - mT, \mathbf{x}_{m+i})\right] dz \\
&= \frac{A^2}{2T} \sum_i \sum_m \int_{-iT}^{-iT+T} \mathrm{E}\left[b^*(z, \mathbf{x}_0) b(z + \tau - mT, \mathbf{x}_m)\right] dz \\
&= \frac{A^2}{2T} \sum_m \int_{-\infty}^{\infty} \mathrm{E}\left[b^*(z, \mathbf{x}_0) b(z + \tau - mT, \mathbf{x}_m)\right] dz \quad ,
\end{aligned}
$$

where $\langle \cdot \rangle$ denotes time averaging and the second last equality used the stationary property of the data sequence $\{\mathbf{x}_k\}$.

- The psd of $\tilde{s}(t)$ is obtained by taking the Fourier transform of $\phi_{\tilde{s}\tilde{s}}(\tau)$,

$$\begin{aligned}
S_{\tilde{s}\tilde{s}}(f) &= \mathrm{E}\left[\frac{A^2}{2T}\sum_m \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} b^*(z,\mathbf{x}_0)b(z+\tau-mT,\mathbf{x}_m)dz\mathrm{e}^{-j2\pi f\tau}d\tau\right] \\
&= \mathrm{E}\left[\frac{A^2}{2T}\sum_m \int_{-\infty}^{\infty} b^*(z,\mathbf{x}_0)\mathrm{e}^{j2\pi fz}dz\right. \\
&\quad \left.\times\int_{-\infty}^{\infty} b(z+\tau-mT,\mathbf{x}_m)\mathrm{e}^{-j2\pi f(z+\tau-mT)}d\tau\mathrm{e}^{-j2\pi fmT}\right] \\
&= \mathrm{E}\left[\frac{A^2}{2T}\sum_m \int_{-\infty}^{\infty} b^*(z,\mathbf{x}_0)\mathrm{e}^{-j2\pi fz}dz\int_{-\infty}^{\infty} b(\tau',\mathbf{x}_m)\mathrm{e}^{-j2\pi f\tau'}d\tau'\mathrm{e}^{-j2\pi fmT}\right] \\
&= \frac{A^2}{2T}\sum_m \mathrm{E}\left[B^*(f,\mathbf{x}_0)B(f,\mathbf{x}_m)\right]\mathrm{e}^{-j2\pi fmT} \quad ,
\end{aligned}$$

where $B(f,\mathbf{x}_m)$ is the Fourier transform of $b(t,\mathbf{x}_m)$.

- Finally,

$$S_{\tilde{s}\tilde{s}}(f) = \frac{A^2}{T}\sum_m S_{b,m}(f)e^{-j2\pi fmT}$$

where

$$S_{b,m}(f) = \frac{1}{2}\mathrm{E}\left[B^*(f,\mathbf{x}_0)B(f,\mathbf{x}_m)\right]$$

- Suppose that $\mathbf{x}_m$ and $\mathbf{x}_0$ are uncorrelated for $|m| \geq K$.
- Then

$$S_{b,m}(f) = S_{b,K}(f), \quad |m| \geq K$$

where

$$\begin{aligned} S_{b,K}(f) &= \frac{1}{2}\mathrm{E}\left[B^*(f,\mathbf{x}_0)\right]\mathrm{E}\left[B(f,\mathbf{x}_m)\right] \quad |m| \geq K \\ &= \frac{1}{2}\mathrm{E}\left[B^*(f,\mathbf{x}_0)\right]\mathrm{E}\left[B(f,\mathbf{x}_0)\right] \quad |m| \geq K \\ &= \frac{1}{2}\left|\mathrm{E}\left[B(f,\mathbf{x}_0)\right]\right|^2 \ , \quad |m| \geq K \ . \end{aligned}$$

- It follows that

$$S_{\tilde{s}\tilde{s}}(f) = S^c_{\tilde{s}\tilde{s}}(f) + S^d_{\tilde{s}\tilde{s}}(f)$$

where

$$\begin{aligned} S^c_{\tilde{s}\tilde{s}}(f) &= \frac{A^2}{T}\sum_{|m|<K}\left(S_{b,m}(f) - S_{b,K}(f)\right)e^{-j2\pi f m T} \\ S^d_{\tilde{s}\tilde{s}}(f) &= \left(\frac{A}{T}\right)^2 S_{b,K}(f)\sum_n \delta\left(f - \frac{n}{T}\right) \end{aligned}$$

- Note that the spectrum consists of discrete and continuous parts. The discrete portion has spectral lines spaced at $1/T$ Hz apart.

ZERO MEAN SIGNALS

- If $\tilde{s}(t)$ has zero mean, i.e., $\mathrm{E}[b(t, \underline{\mathbf{x}}_0)] = 0$, then $\mathrm{E}[B(f, \underline{\mathbf{x}}_0)] = 0$.
- Under this condition

$$S_{b,K}(f) = \frac{1}{2} \left| \mathrm{E}[B(f, \underline{\mathbf{x}}_0)] \right|^2 = 0$$

- Hence, $S_{\tilde{s}\tilde{s}}(f)$ has no discrete component and

$$S_{\tilde{s}\tilde{s}}(f) = \left(\frac{A^2}{T} \right) \sum_{|m|<K} S_{b,m}(f) e^{-j2\pi f m T}$$

UNCORRELATED SOURCE SYMBOLS

- With uncorrelated source symbols the information symbols $x_{m,k}$ constituting data blocks $\mathbf{x}_m = (x_{m,1}, x_{m,2}, \ldots, x_{m,N})$ are mutually uncorrelated. Under this condition $\mathbf{x}_m$ and $\mathbf{x}_0$ are obviously uncorrelated for $|m| \geq 1$.

- Hence, $S_{b,m}(f) = S_{b,1}(f)$, for $|m| \geq 1$, where

$$\begin{aligned} S_{b,0}(f) &= \frac{1}{2}\mathrm{E}\left[|B(f,\mathbf{x}_0)|^2\right] \\ S_{b,1}(f) &= \frac{1}{2}\left|\mathrm{E}\left[B(f,\mathbf{x}_0)\right]\right|^2 \end{aligned}$$

- Hence

$$\begin{aligned} S^d_{\tilde{s}\tilde{s}}(f) &= \frac{A^2}{T^2}S_{b,1}(f)\sum_n \delta(f - n/T) \\ S^c_{\tilde{s}\tilde{s}}(f) &= \frac{A^2}{T}\left(S_{b,0}(f) - S_{b,1}(f)\right) \end{aligned}$$

- If $\tilde{s}(t)$ has zero mean as well, then

$$S_{\tilde{s}\tilde{s}}(f) = \frac{A^2}{T}S_{b,0}(f)$$

LINEAR FULL RESPONSE MODULATION

- Here it is assumed that

$$\begin{aligned} b(t, \mathbf{x}_k) &= x_k h_a(t) \\ B(f, \mathbf{x}_k) &= x_k H_a(f) \ , \end{aligned}$$

  where the x_k may be correlated.

- Using the above leads to

$$\begin{aligned} S_{b,m}(f) &= \frac{1}{2}\mathrm{E}\left[B^*(f, \mathbf{x}_0) B(f, \mathbf{x}_m)\right] \\ &= \frac{1}{2}\mathrm{E}\left[x_0^* H_a^*(f)) x_m H_a(f))\right] \\ &= \frac{1}{2}\mathrm{E}\left[x_0^* x_m \left|H_a(f)\right|^2\right] \\ &= \frac{1}{2}\mathrm{E}\left[x_0^* x_m\right] \left|H_a(f)\right|^2 \\ &= \phi_{xx}(m) \left|H_a(f)\right|^2 \end{aligned}$$

  where

$$\phi_{xx}(m) = \frac{1}{2}\mathrm{E}[x_k^* x_{k+m}]$$

- The psd of the complex envelope is

$$\begin{aligned} S_{\tilde{s}\tilde{s}}(f) &= \frac{A^2}{T} \sum_m S_{b,m}(f) \mathrm{e}^{-j2\pi f m T} \\ &= \frac{A^2}{T} |H_a(f)|^2 \sum_m \phi_{xx}(m) \mathrm{e}^{-j2\pi f m T} \\ &= \frac{A^2}{T} |H_a(f)|^2 S_{xx}(f) \end{aligned}$$

where

$$S_{xx}(f) = \sum_m \phi_{xx}(m) e^{-j2\pi f m T}$$

- With uncorrelated source symbols

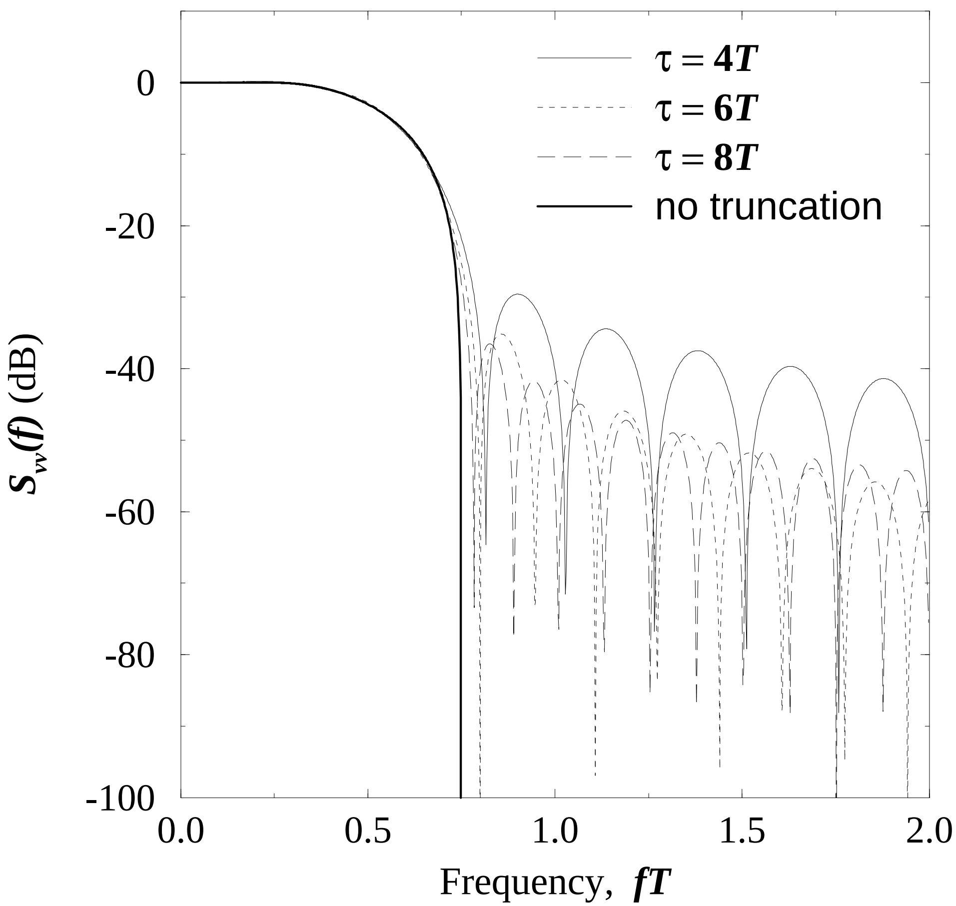
$$\begin{aligned} S_{b,0}(f) &= \sigma_x^2 |H_a(f)|^2 \\ S_{b,m}(f) &= \frac{1}{2} |\mu_x|^2 |H_a(f)|^2 \ , \quad |m| \geq 1 \ . \end{aligned}$$

where $\sigma_x^2 = \frac{1}{2}\mathrm{E}[|x_k|^2]$, $\mu_x = \mathrm{E}[x_k]$.

- If $\mu_x = 0$, then $S_{b,1}(f) = 0$ and

$$S_{\tilde{s}\tilde{s}}(f) = \frac{A^2}{T} \sigma_x^2 |H_a(f)|^2$$

POWER SPECTRAL DENSITY OF ASK



Psd of ASK with a truncated square root raised cosine pulse with various truncation lengths; $\beta = 0.5$.

OFDM Power Spectrum

- The data symbols $x_{n,k}, k = 0, \ldots, N-1$ that modulate the N sub-carriers are assumed to have zero mean, variance $\sigma_x^2 = \frac{1}{2}\mathrm{E}[|x_{n,k}|^2]$, and they are mutually uncorrelated.
- In this case, the psd of the OFDM waveform is

$$S_{\tilde{s}\tilde{s}}(f) = \frac{A^2}{T_g} S_{b,0}(f) \ ,$$

where

$$S_{b,0}(f) = \frac{1}{2}\mathrm{E}\left[|B(f, \mathbf{x}_0)|^2\right] \ ,$$

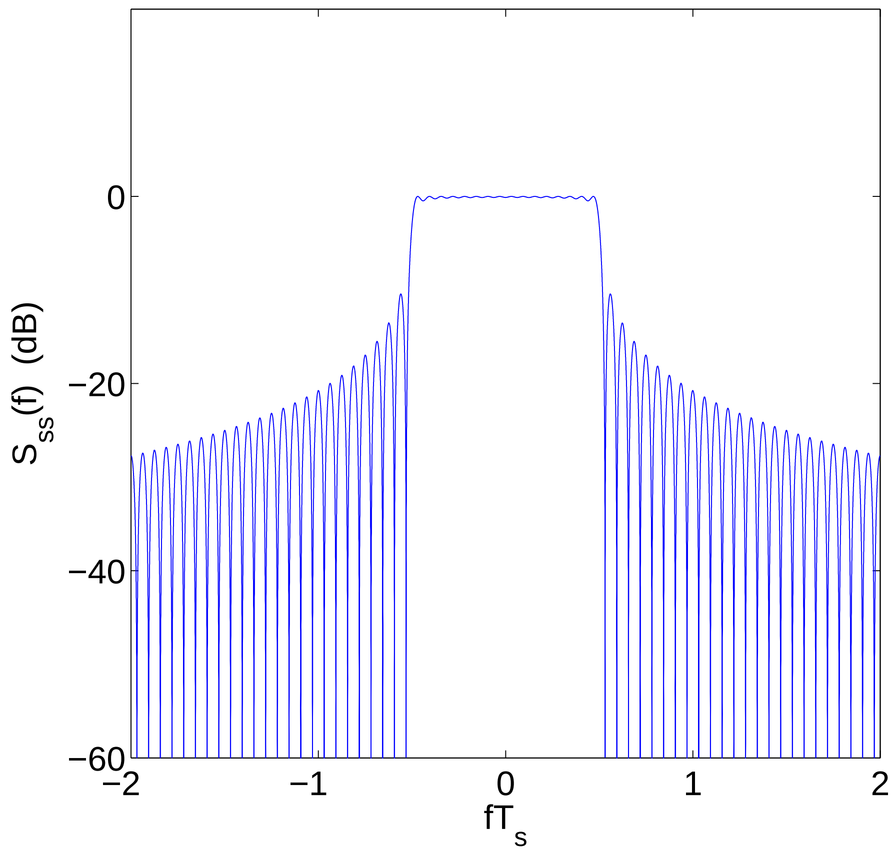
and

$$B(f, \mathbf{x}_0) = \sum_{k=0}^{N-1} x_{0,k} T \mathrm{sinc}(fT - k) + \sum_{k=0}^{N-1} x_{0,k} \alpha_g T \mathrm{sinc}(\alpha_g(fT - k)) e^{j2\pi fT} \ .$$

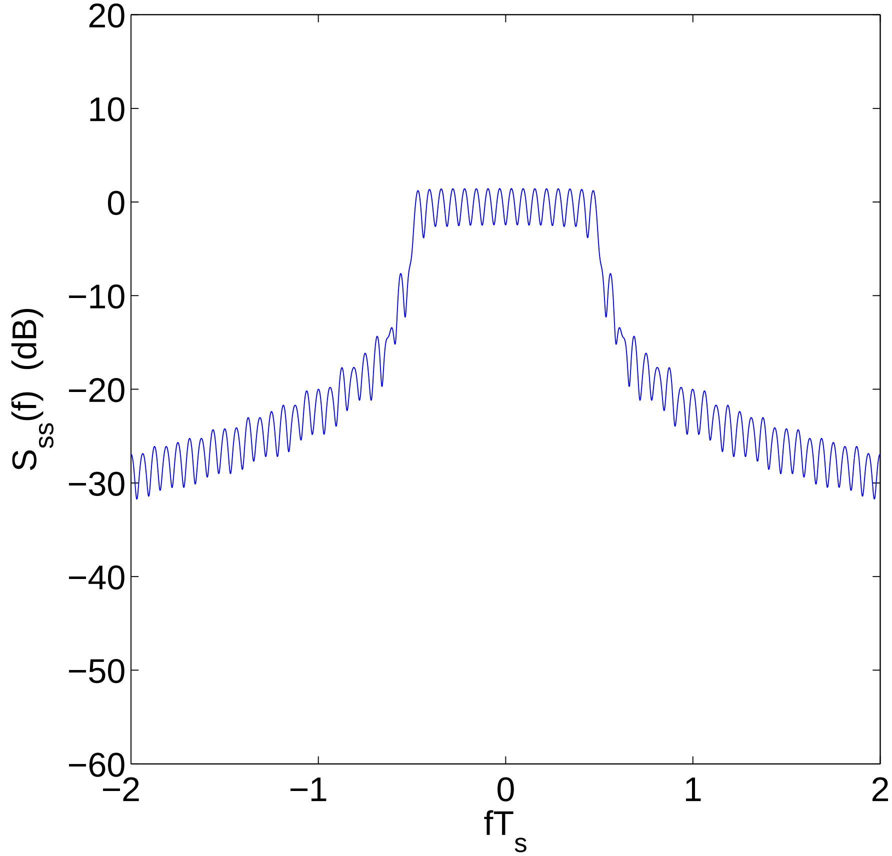
Using the above along with $T = NT_s$ yields the result

$$\begin{aligned} S_{\tilde{s}\tilde{s}}(f) \ &= \ \sigma_x^2 A^2 T \left(\frac{1}{1+\alpha_g} \sum_{k=0}^{N-1} \mathrm{sinc}^2(NfT_s - k) \right. \\ &\quad + \frac{\alpha_g^2}{1+\alpha_g} \sum_{k=0}^{N-1} \mathrm{sinc}^2(\alpha_g(NfT_s - k)) \\ &\quad \left. + \frac{2\alpha_g}{1+\alpha_g} \cos(2\pi NfT_s) \sum_{k=0}^{N-1} \mathrm{sinc}(NfT_s - k) \mathrm{sinc}(\alpha_g(NfT_s - k)) \right) \ . \end{aligned}$$

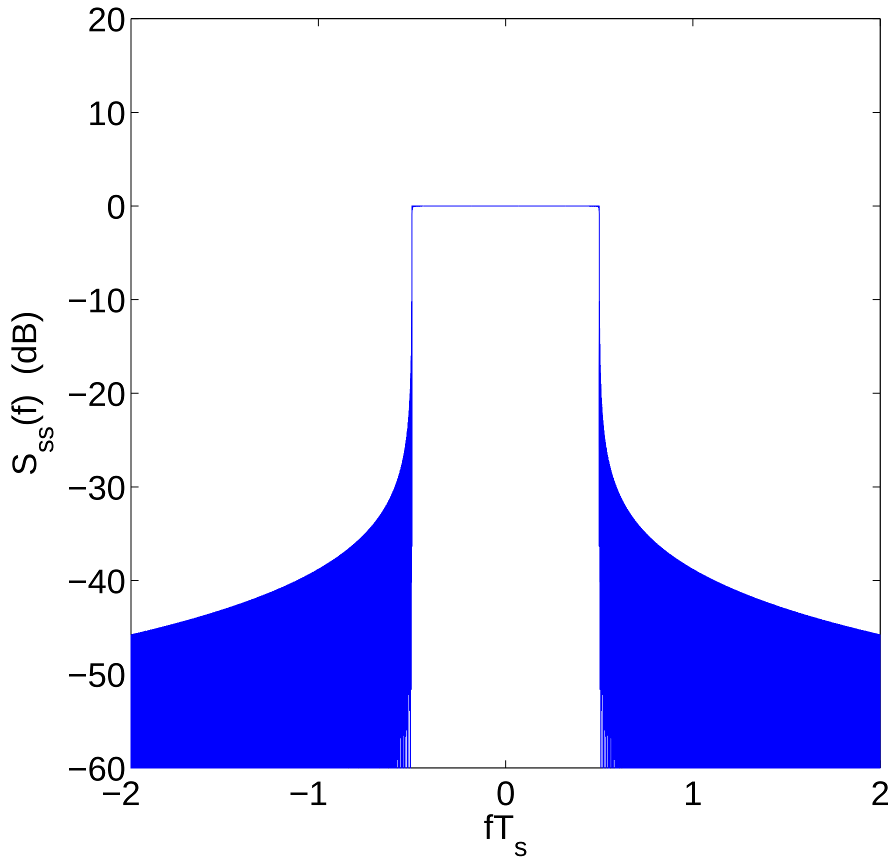
- Note that the Nyquist frequency in this case is $1/2T_s^g = (1+\alpha_g)/2T_s$.



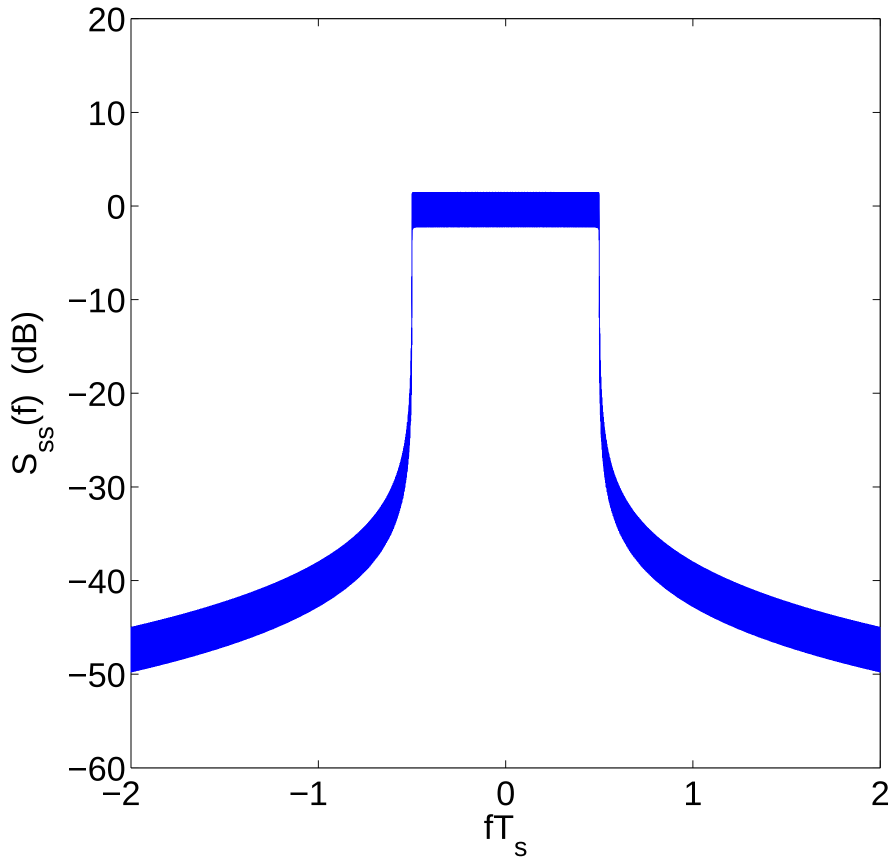
Psd of OFDM with $N = 16, \alpha_g = 0.$



Psd of OFDM with $N = 16, \alpha_g = 0.25$.



Psd of OFDM with $N = 1024, \alpha_g = 0.$



Psd of OFDM with $N = 1024, \alpha_g = 0.25.$

OFDM Power Spectrum -IFFT Implementation

- The output of the IDFT baseband modulator is $\{\mathbf{X}^g\} = \{X^g_{n,m}\}$, where m is the block index and

$$
\begin{aligned}
X^g_{n,m} &= X_{n,(m)_N} \\
&= A \sum_{k=0}^{N-1} x_{n,k} e^{\frac{j2\pi km}{N}} \;, \quad m = 0,\ 1,\ \ldots,\ N+G-1
\end{aligned}
$$

- The power spectrum of the sequence $\{\mathbf{X}^g\}$ can be calculated by first determining the discrete-time autocorrelation function of the time-domain sequence $\{\mathbf{X}^g\}$ and then taking a discrete-time Fourier transform of the discrete-time autocorrelation function.
- The psd of the OFDM complex envelope with ideal DACs can be obtained by applying the resulting power spectrum to an ideal low-pass filter with a cutoff frequency of $1/(2T^g_s)$ Hz.

Discrete-time Autocorrelation Function

- The time-domain sequence $\{\mathbf{X}^g\}$ is a periodic wide-sense stationary sequence having the discrete-time autocorrelation function

$$\begin{aligned}\phi_{X^g X^g}(m,\ell) &= \frac{1}{2}\mathrm{E}[(X^g_{n,m})^* X^g_{n,m+\ell}] \\ &= A^2 \sum_{k=0}^{N-1}\sum_{i=0}^{N-1}\frac{1}{2}\mathrm{E}[x^*_{n,k}x_{n,i}]e^{j\frac{2\pi}{N}(-km+im+i\ell)}, \\ &\qquad \text{for } m = 0,\ldots,N+G-1 \ .\end{aligned}$$

The data symbols, $x_{n,k}$, are assumed to be mutually uncorrelated with zero mean and variance $\sigma_x^2 = \frac{1}{2}\mathrm{E}[|x_{n,k}|^2]$. Using the fact, that $X^g_{n,m} = X_{n,(m)_N}$, it follows that

$$\phi_{X^g X^g}(m,\ell) = \begin{cases} & m = 0,\ldots,G-1, \ell = 0, N \\ A\sigma_x^2 & m = G,\ldots,N-1, \ell = 0 \\ & m = N,\ldots,N+G-1, \ell = 0, -N \\ 0 & \text{otherwise} \end{cases} .$$

Averaging over all time indices m in a block gives the time-average discrete-time autocorrelation function

$$\phi_{X^g X^g}(\ell) = \begin{cases} A\sigma_x^2 & \ell = 0 \\ \frac{G}{N+G}A\sigma_x^2 & \ell = -N, N \\ 0 & \text{otherwise} \end{cases} .$$

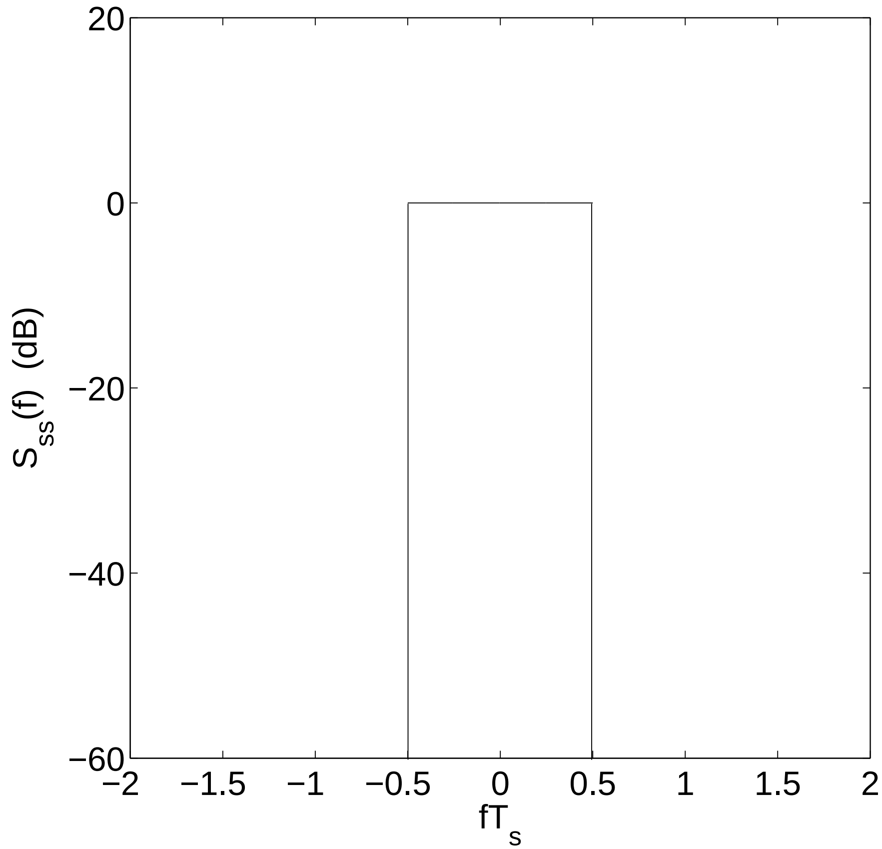
Power Spectrum

- Taking the discrete-time Fourier transform of the discrete-time autocorrelation function gives

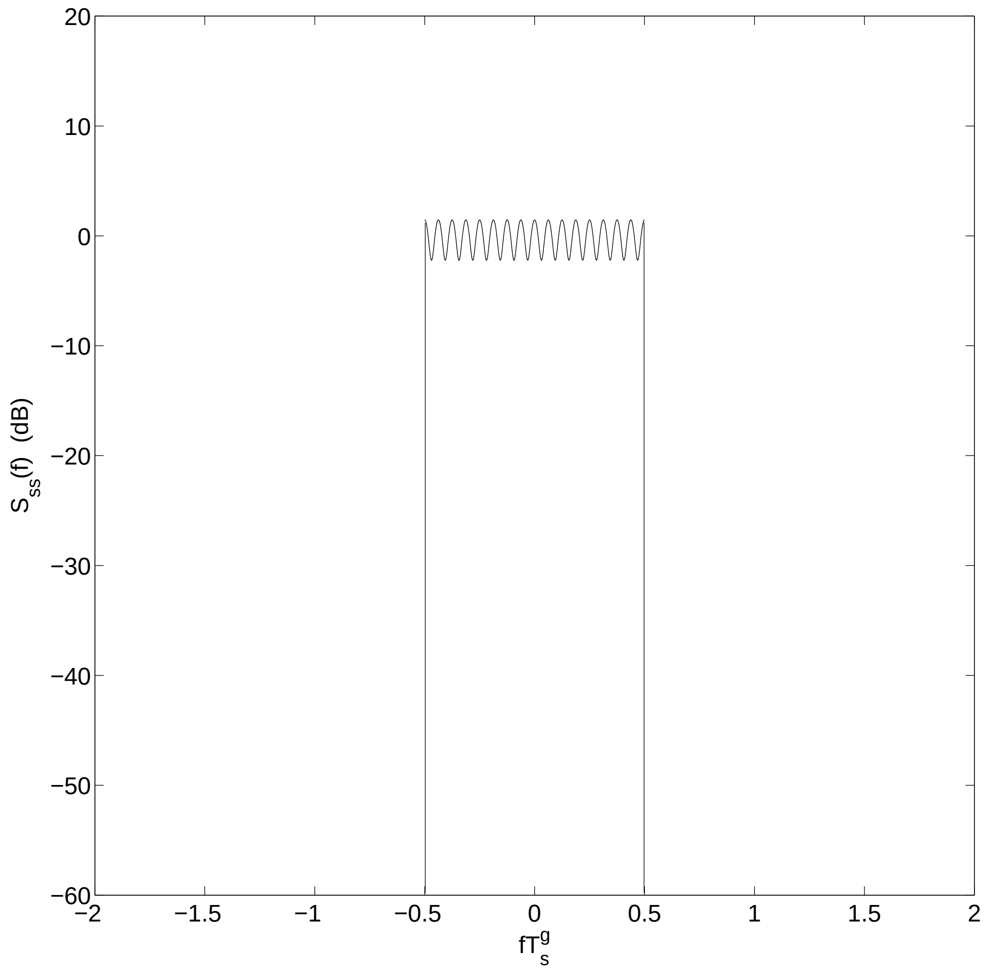
$$
\begin{aligned}
S_{X^g X^g}(f) &= \sum_m \phi_{X^g X^g}(\ell) e^{-j2\pi f m N T_s^g} \\
&= A\sigma_x^2 \left(1 + \frac{G}{N+G} e^{-j2\pi f N T_s^g} + \frac{G}{N+G} e^{j2\pi f N T_s^g}\right) \\
&= A\sigma_x^2 \left(1 + \frac{2G}{N+G} \cos(2\pi f N T_s^g)\right) \quad .
\end{aligned}
$$

- Finally, assume that the sequence $\{\mathbf{X}^g\} = \{X_{n,m}^g\}$ is passed through an ideal DACs.
  - The ideal DAC is a low-pass filter with cutoff frequency $1/(2T_s^g)$.
- The OFDM complex envelope has the psd

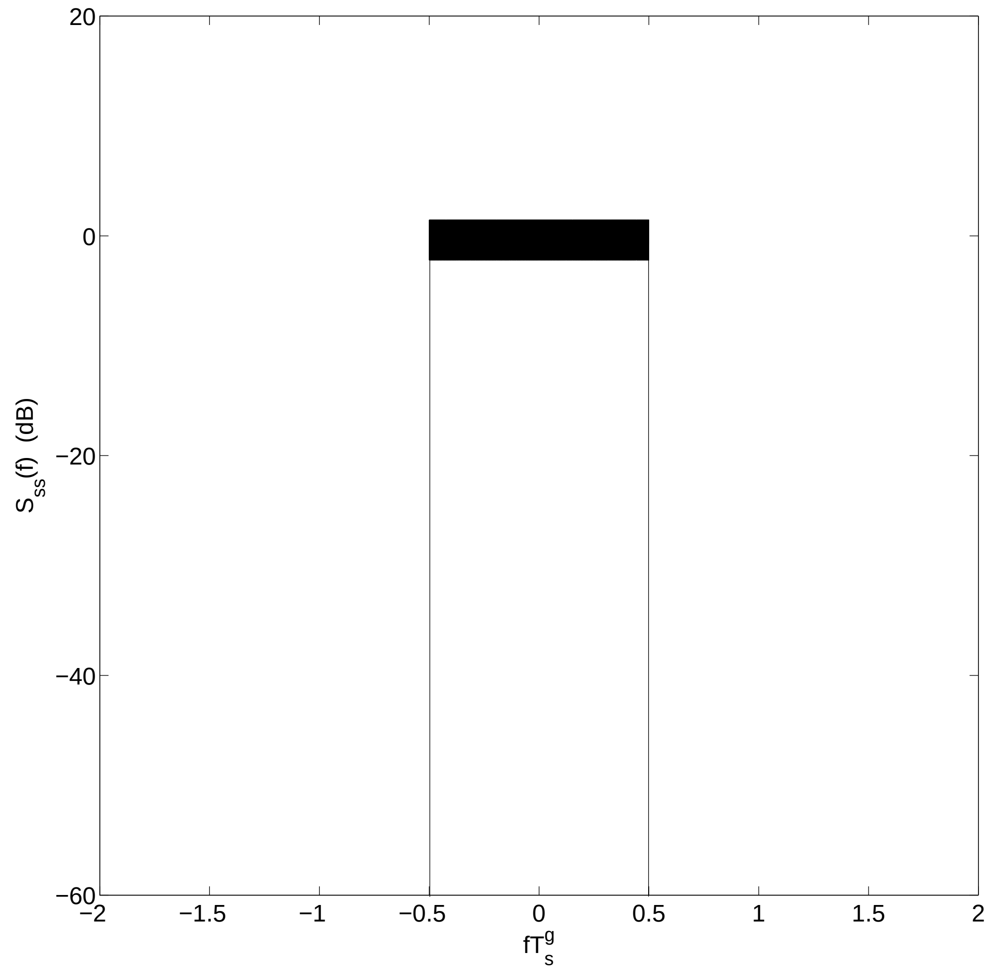
$$
S_{\tilde{s}\tilde{s}}(f) = A\sigma_x^2 \left(1 + \frac{2G}{N+G} \cos(2\pi f N T_s^g)\right) \mathrm{rect}\left(f T_s^g\right) \quad .
$$



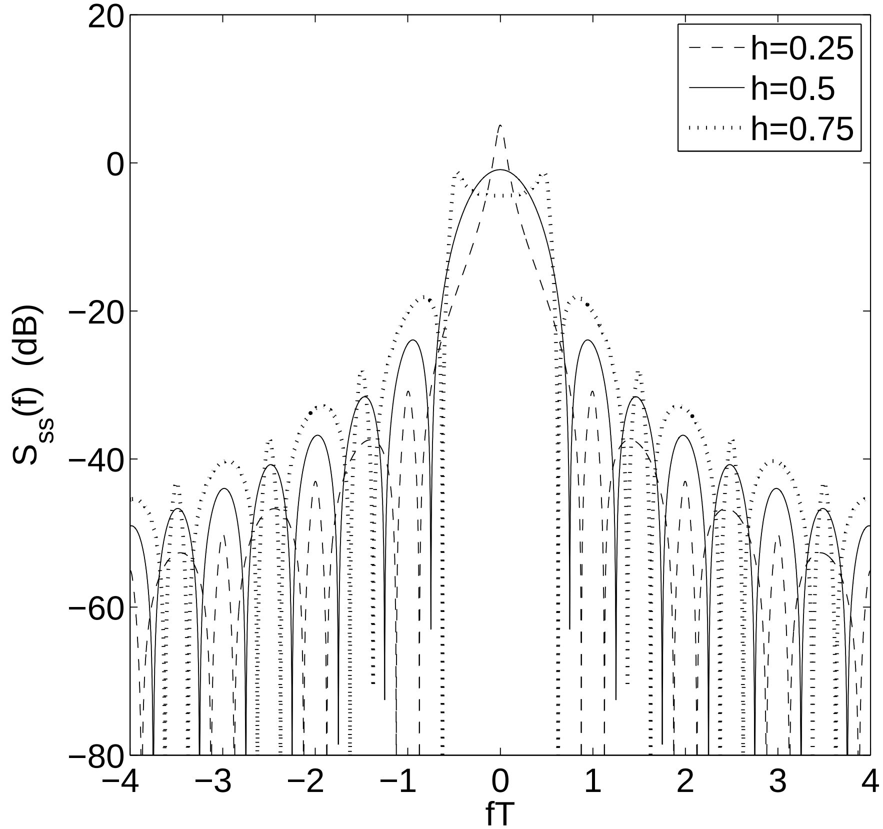
Psd of IDFT-based OFDM with $N = 16, G = 0$. *Note in this case that* $T_s^g = T_s$.



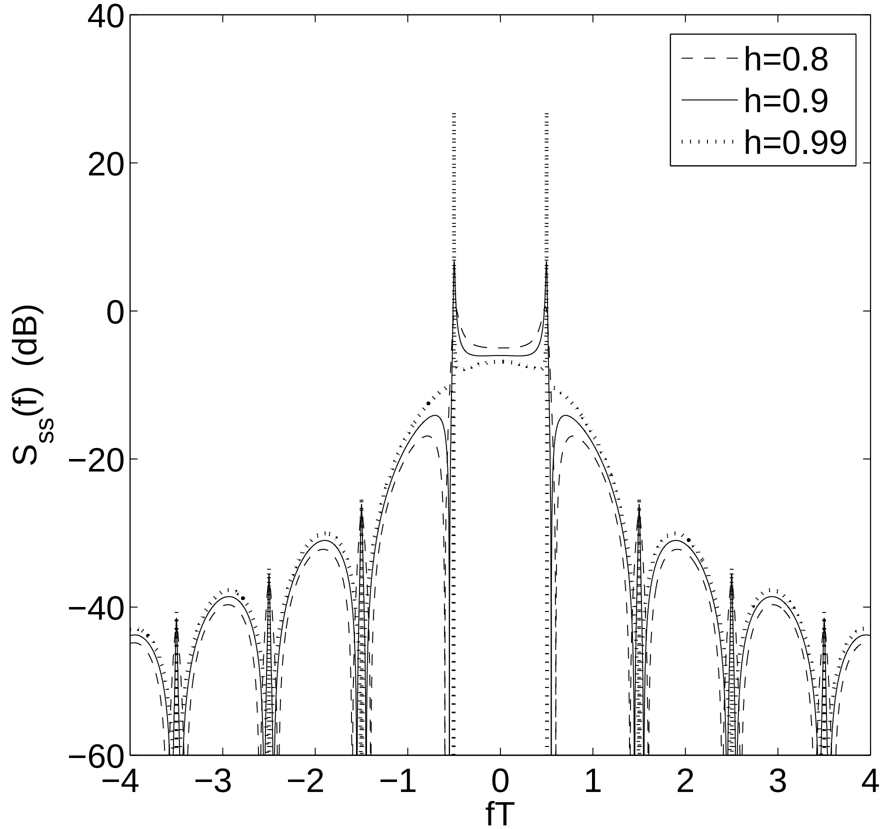
Psd of IDFT-based OFDM with $N = 16, G = 4.$



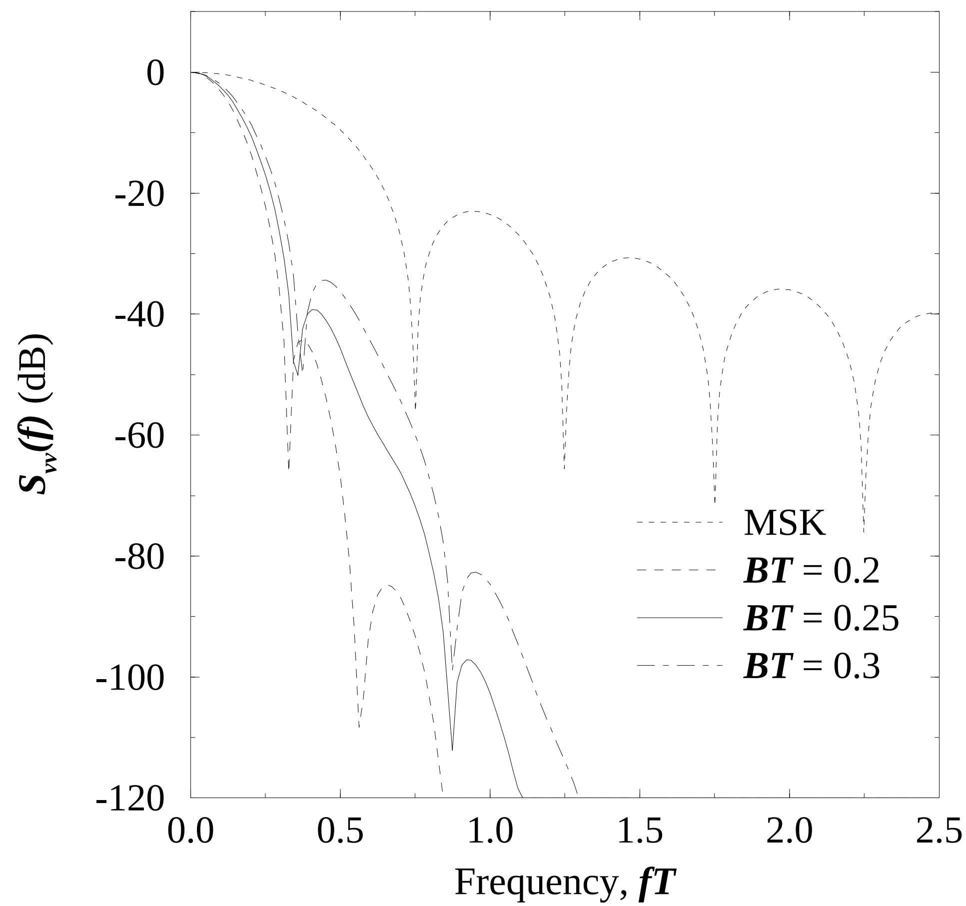
Psd of IDFT-based OFDM with $N = 1024, G = 256$.



Power spectral density of binary CPFSK for various modulation indices.



Psd of binary CPFSK as the modulation index $h \to 1$.



Power spectral density of GMSK with various normalized filter bandwidths BT.