

**EE6604**

**Personal & Mobile Communications**

Week 13

Multi-antenna Techniques

Reading: 6.1 - 6.8

# Diversity Methods

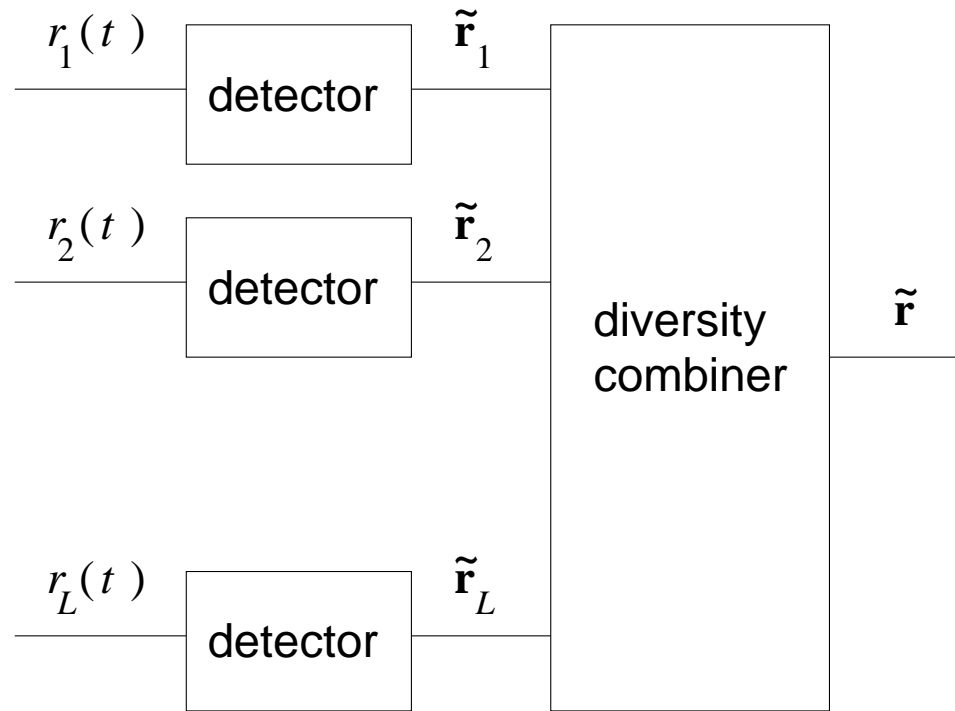
- Diversity combats fading by providing the receiver with multiple uncorrelated replicas of the same information bearing signal.
- There are several types of **receiver diversity** methods:
  - **spatial**, angle, polarization, frequency, time, and multipath diversity
- There are different methods to combine the diversity branches.
  - **maximal ratio, equal gain**, switched, and selective combining.
- If the signal  $\tilde{s}_m(t)$  is transmitted, the received complex envelopes on the different diversity branches are

$$\tilde{r}_k(t) = g_k \tilde{s}_m(t) + \tilde{n}_k(t), \quad k = 1, \dots, L$$

- $L$  is the number of diversity branches.
  - $g_k = \alpha_k e^{j\phi_k}$  is the fading gain associated with the  $k^{\text{th}}$  branch.
  - The AWGN processes  $\tilde{n}_k(t)$  are independent from branch to branch.
- With Rayleigh fading, the instantaneous received modulated symbol energy-to-noise ratio on the  $k$ th diversity branch has the exponential pdf

$$p_{\gamma_k}(x) = \frac{1}{\bar{\gamma}_k} e^{-x/\bar{\gamma}_k},$$

where  $\bar{\gamma}_k$  is the average received branch symbol energy-to-noise ratio for the  $k$ th diversity branch. For balanced diversity branches  $\bar{\gamma}_k = \bar{\gamma}_c$ ,  $k = 1, \dots, L$ .



*Diversity Combining.*

## Selective Combining (SC)

- With ideal SC, the branch with the largest symbol energy-to-noise ratio is always selected such that

$$\tilde{\mathbf{r}} = \max_{|g_k|} \tilde{\mathbf{r}}_k .$$

- The instantaneous symbol energy-to-noise ratio at the output of the selective combiner is

$$\gamma_s^s = \max \{ \gamma_1, \gamma_2, \dots, \gamma_L \} ,$$

where  $L$  is the number of branches.

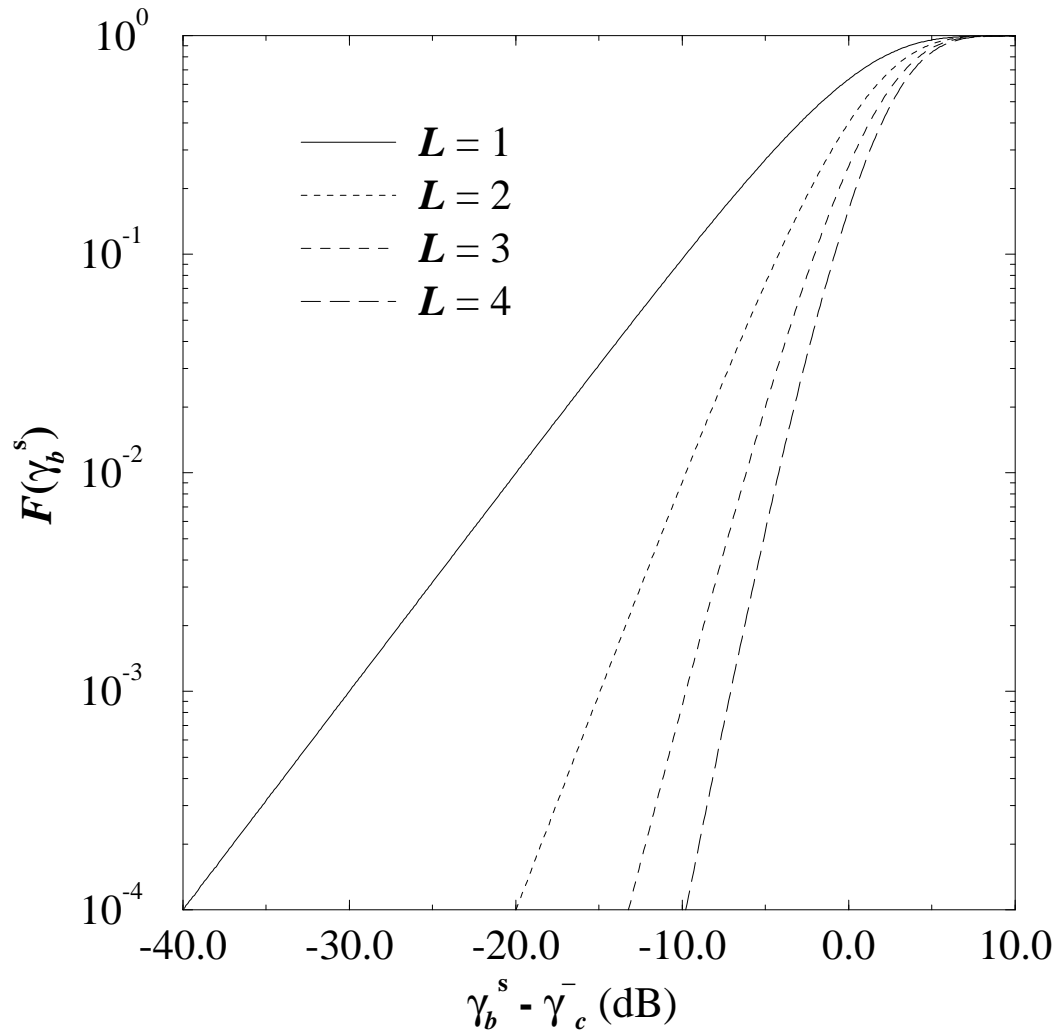
- If the branches are independently faded, then order statistics gives the cumulative distribution function (cdf)

$$F_{\gamma_s^s}(x) = \Pr [\gamma_1 \leq x, \gamma_2 \leq x, \dots, \gamma_L \leq x] = [1 - e^{-x/\bar{\gamma}_c}]^L , \quad x \geq 0 .$$

- Differentiating the above expression gives the pdf of the instantaneous output symbol energy-to-noise ratio as

$$p_{\gamma_s^s}(x) = \frac{L}{\bar{\gamma}_c} [1 - e^{-x/\bar{\gamma}_c}]^{L-1} e^{-x/\bar{\gamma}_c} , \quad x \geq 0 .$$

- The above pdf can be used to evaluate the performance of various digital modulation schemes.



*Cdf of  $\gamma_b^s$  for selective combining;  
 $\bar{\gamma}_c$  is the average branch symbol energy-to-noise ratio.*

# Error Probability with Selective Combining (SC)

- The bit error probability with slow flat fading can be obtained by averaging the bit error probability, as a function of the symbol energy-to-noise ratio, over the pdf of  $\gamma_s^s$ . For example, the bit error probability for binary DPSK with differential detection on an AWGN channel is

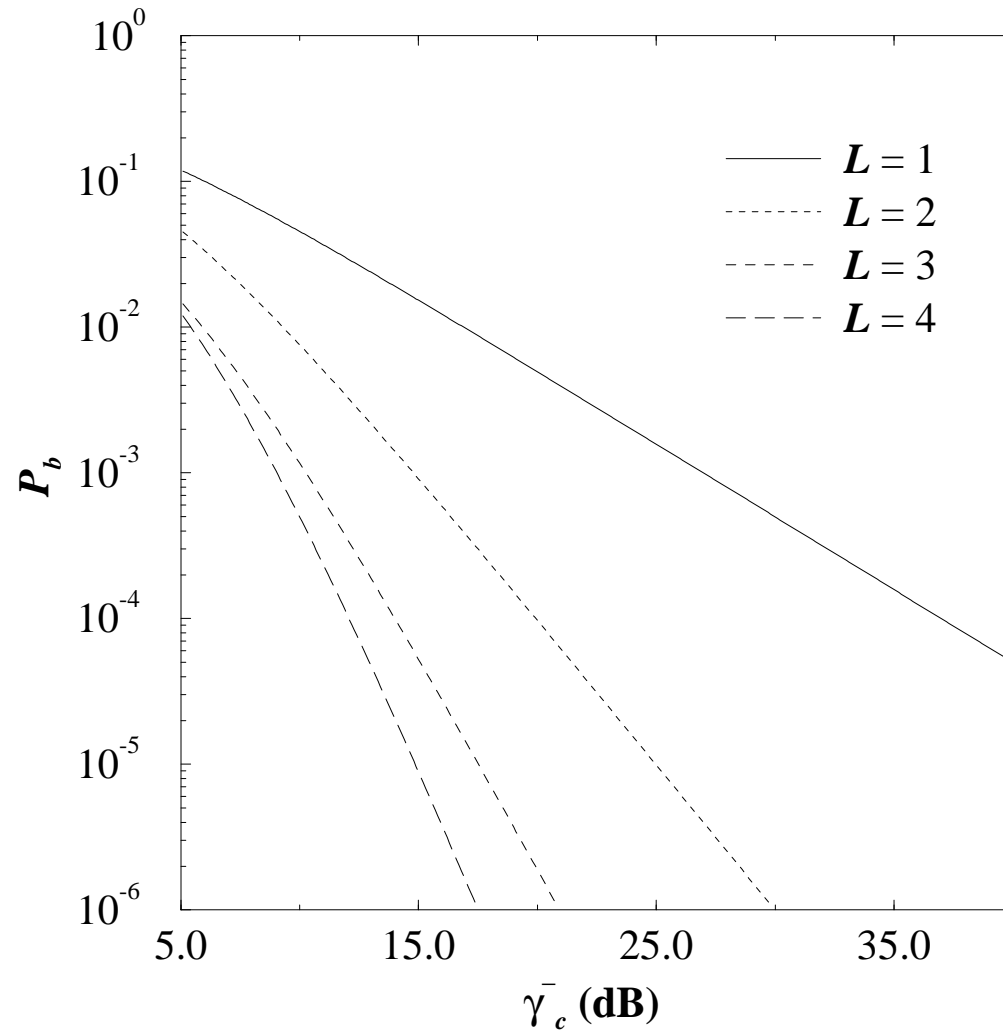
$$P_b(\gamma_s^s) = \frac{1}{2}e^{-\gamma_s^s} ,$$

where  $\gamma_s^s$  can be interpreted as the instantaneous bit energy-to-noise ratio since binary modulation is being used. Hence, with SC

$$\begin{aligned} P_b &= \int_0^\infty P_b(x)p_{\gamma_s^s}(x)dx \\ &= \int_0^\infty \frac{L}{2\bar{\gamma}_c}e^{-(1+1/\bar{\gamma}_c)x} \left(1 - e^{-x/\bar{\gamma}_c}\right)^{L-1} dx \\ &= \frac{L}{2\bar{\gamma}_c} \sum_{n=0}^{L-1} \binom{L-1}{n} (-1)^n \int_0^\infty e^{-(1+(n+1)/\bar{\gamma}_c)x} dx \\ &= \frac{L}{2} \sum_{n=0}^{L-1} \frac{\binom{L-1}{n} (-1)^n}{1+n+\bar{\gamma}_c} , \end{aligned}$$

where we have used the binomial expansion

$$(1-x)^{L-1} = \sum_{n=0}^{L-1} \binom{L-1}{n} (-1)^n x^n .$$



*Bit error probability for binary DPSK with differential detection and  $L$ -branch selective diversity combining.*

# Error Probability with Selective Combining (SC)

- The bit error probability for BPSK on an AWGN channel is

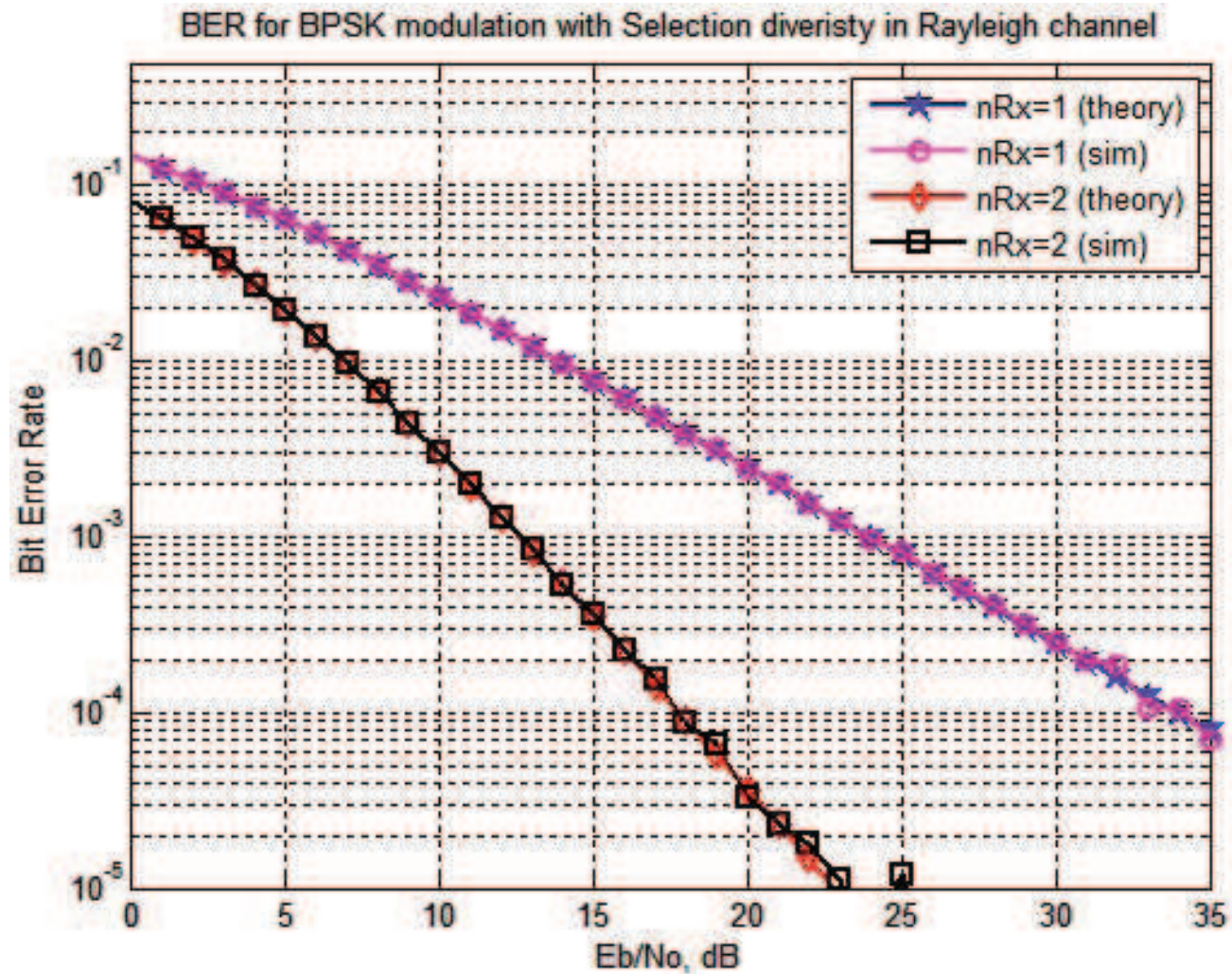
$$P_b(\gamma_s^s) = Q(\sqrt{2\gamma_b^s}) \quad ,$$

where  $\gamma_b^s$  is the instantaneous bit energy-to-noise ratio at the output of the selective combiner. Hence, with selective combining

$$\begin{aligned} P_b &= \int_0^\infty P_b(x) p_{\gamma_b^s}(x) dx \\ &= \int_0^\infty Q(\sqrt{2x}) \frac{L}{\bar{\gamma}_c} [1 - e^{-x/\bar{\gamma}_c}]^{L-1} e^{-x/\bar{\gamma}_c} dx \\ &= \frac{1}{2} \sum_{k=0}^{L-1} (-1)^k \binom{L-1}{k} \left(1 + \frac{k}{\bar{\gamma}_c}\right)^{-1/2} \end{aligned}$$

where  $\bar{\gamma}_c$  is the average branch bit energy-to-noise ratio.





*Bit error probability for BPSK and 1 and 2-branch selective diversity combining.*

# Maximal Ratio Combining (MRC)

- The vector

$$\tilde{\mathbf{r}} = \text{vec}(\tilde{\mathbf{r}}_1, \tilde{\mathbf{r}}_2, \dots, \tilde{\mathbf{r}}_L)$$

has the multivariate complex Gaussian distribution

$$p(\tilde{\mathbf{r}}|\mathbf{g}, \tilde{\mathbf{s}}_m) = \frac{1}{(2\pi N_o)^{LN}} \exp\left\{-\frac{1}{2N_o} \sum_{k=1}^L \|\tilde{\mathbf{r}}_k - g_k \tilde{\mathbf{s}}_m\|^2\right\}$$

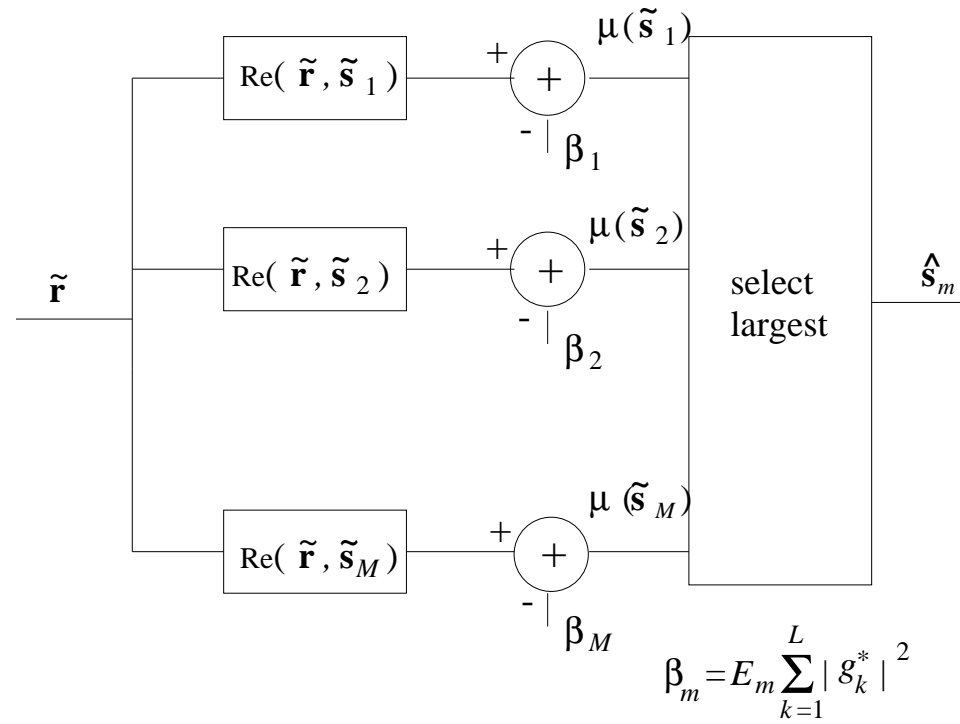
where  $\mathbf{g} = (g_1, g_2, \dots, g_L)$  is the channel vector.

- The maximum likelihood receiver chooses the message vector  $\tilde{\mathbf{s}}_m$  to maximize the likelihood function  $p(\tilde{\mathbf{r}}|\mathbf{g}, \tilde{\mathbf{s}}_m)$  or the log-likelihood function  $\log p(\tilde{\mathbf{r}}|\mathbf{g}, \tilde{\mathbf{s}}_m)$ . This is equivalent to choosing  $\tilde{\mathbf{s}}_m$  to minimize the decision variable

$$\begin{aligned} \mu(\tilde{\mathbf{s}}_m) &= \sum_{k=1}^L \|\tilde{\mathbf{r}}_k - g_k \tilde{\mathbf{s}}_m\|^2 \\ &= \sum_{k=1}^L \left( \|\tilde{\mathbf{r}}_k\|^2 - 2\text{Re}\{\tilde{\mathbf{r}}_k \cdot g_k^* \tilde{\mathbf{s}}_m^*\} + |g_k|^2 \|\tilde{\mathbf{s}}_m\|^2 \right) \end{aligned}$$

- Alternatively, we can just choose  $\tilde{\mathbf{s}}_m$  to maximize the decision variable

$$\begin{aligned} \mu_2(\tilde{\mathbf{s}}_m) &= \sum_{k=1}^L \text{Re}\{g_k^* \tilde{\mathbf{r}}_k \cdot \tilde{\mathbf{s}}_m^*\} - E_m \sum_{k=1}^L |g_k|^2 \\ &= \text{Re}\left\{ \sum_{k=1}^L g_k^* \tilde{\mathbf{r}}_k \cdot \tilde{\mathbf{s}}_m^* \right\} - E_m \sum_{k=1}^L |g_k|^2 \end{aligned}$$



*Metric computer for maximal ratio combining.*

# MRC Performance

- The MRC receiver combines the diversity branches according to

$$\tilde{\mathbf{r}} = \sum_{k=1}^L g_k^* \tilde{\mathbf{r}}_k$$

and chooses  $\tilde{\mathbf{s}}_m$  to maximize the decision variable

$$\mu(\tilde{\mathbf{s}}_m) = \text{Re}(\tilde{\mathbf{r}}, \tilde{\mathbf{s}}_m) - E_m \sum_{k=1}^L |g_k|^2$$

- To evaluate the performance gain with MRC, we note that

$$\begin{aligned} \tilde{\mathbf{r}} &= \sum_{k=1}^L g_k^* (g_k \tilde{\mathbf{s}}_m + \tilde{\mathbf{n}}_k) \\ &= \left( \sum_{k=1}^L \alpha_k^2 \right) \tilde{\mathbf{s}}_m + \sum_{k=1}^L g_k^* \tilde{\mathbf{n}}_k \\ &\equiv \alpha_M^2 \tilde{\mathbf{s}}_m + \tilde{\mathbf{n}}_M, \end{aligned} \tag{1}$$

where  $\alpha_M^2 = \sum_{k=1}^L \alpha_k^2$ ,  $\tilde{\mathbf{n}}_M = \sum_{k=1}^L g_k^* \tilde{\mathbf{n}}_k$  and  $\alpha_k^2 = |g_k|^2$ .

- The first term in (1) is the signal component with average energy  $\frac{1}{2} \text{E}[\alpha_M^4 \|\tilde{\mathbf{s}}_m\|^2] = \alpha_M^4 E_{\text{av}}$ , where  $E_{\text{av}}$  is the average symbol energy in the signal constellation.

## MRC Performance

- The second term is the noise component with variance (per dimension)

$$\sigma_{\tilde{n}_M}^2 = \frac{1}{2N} \mathbb{E}[\|\tilde{\mathbf{n}}_M\|^2] = N_o \sum_{k=1}^L \alpha_k^2 = N_o \alpha_M^2 .$$

- The ratio of the two gives the symbol energy-to-noise ratio

$$\gamma_s^{\text{mr}} = \frac{\frac{1}{2} \mathbb{E}[\alpha_M^4 \|\tilde{\mathbf{s}}_m\|^2]}{\sigma_{\tilde{n}_M}^2} = \frac{\alpha_M^2 E_{\text{av}}}{N_o} = \sum_{k=1}^L \frac{\alpha_k^2 E_{\text{av}}}{N_o} = \sum_{k=1}^L \gamma_k$$

where  $\gamma_k = \alpha_k^2 E_{\text{av}}/N_o$ . Note that  $\gamma_s^{\text{mr}}$  is the sum of the symbol energy-to-noise ratios of the  $L$  diversity branches.

- If the branches are balanced (which is a reasonable assumption with antenna diversity) and uncorrelated, then  $\gamma_b^{\text{mr}}$  has a central chi-square distribution with  $2L$  degrees of freedom

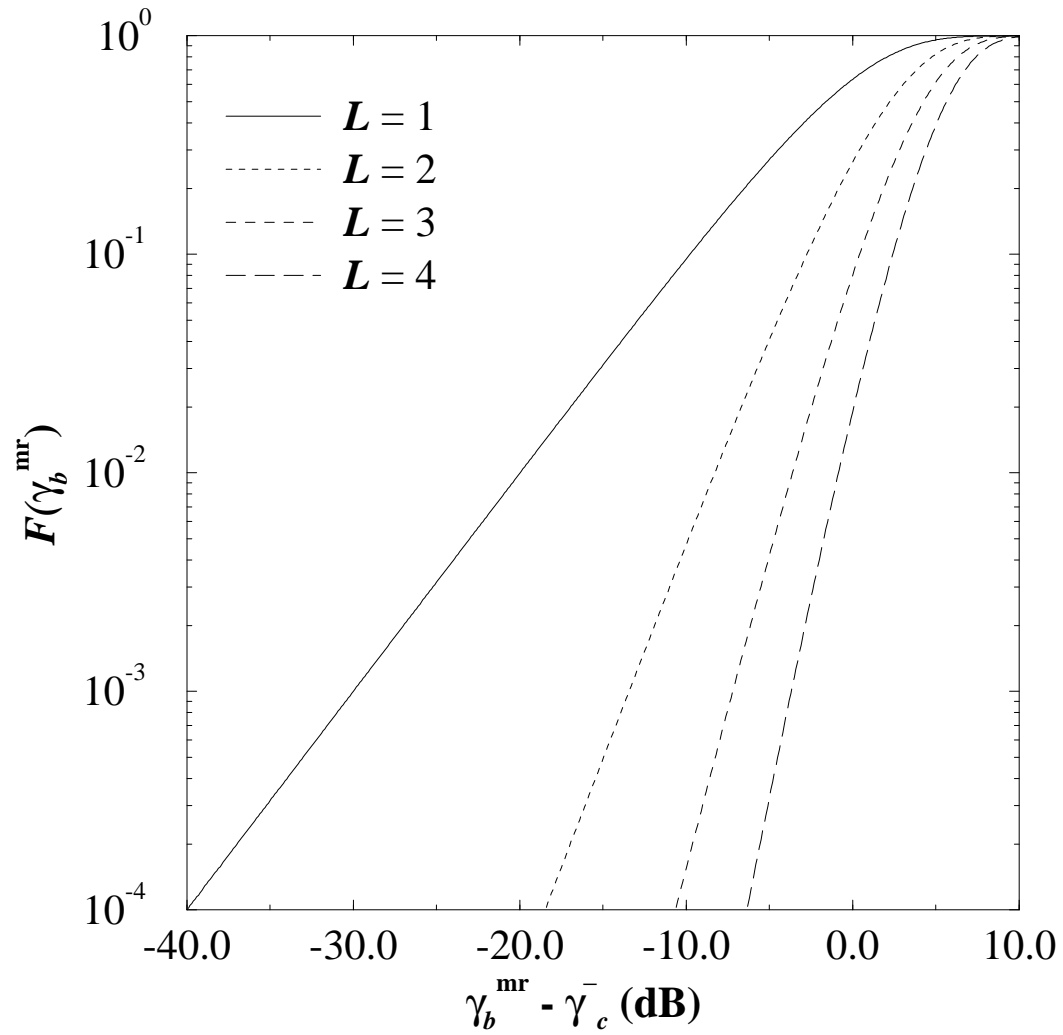
$$p_{\gamma_b^{\text{mr}}}(x) = \frac{1}{(L-1)! (\bar{\gamma}_c)^L} x^{L-1} e^{-x/\bar{\gamma}_c} , \quad x \geq 0$$

where

$$\bar{\gamma}_c = \mathbb{E}[\gamma_k] \quad k = 1, \dots, L$$

- The cdf of  $\gamma_s^{\text{mr}}$  is

$$F_{\gamma_s^{\text{mr}}}(x) = 1 - e^{-x/\bar{\gamma}_c} \sum_{k=0}^{L-1} \frac{1}{k!} \left( \frac{x}{\bar{\gamma}_c} \right)^k , \quad x \geq 0 .$$



*Cdf of  $\gamma_s^{\text{mr}}$  for maximal ratio combining;  
 $\bar{\gamma}_c$  is the average branch symbol energy-to-noise ratio.*

## Performance of BPSK with MRC

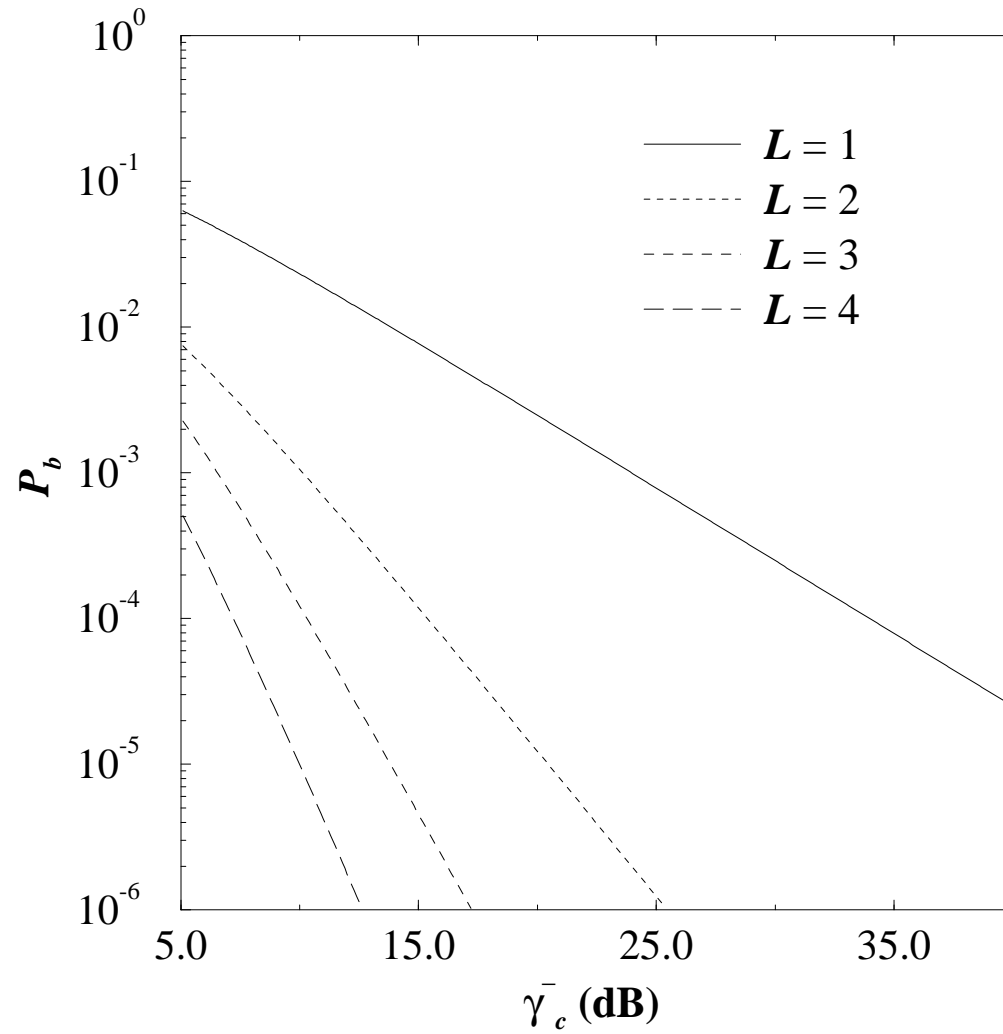
- When computing the probability of bit error, we must limit our attention to coherent signaling techniques since MRC is a coherent detection technique.
- The bit error probability with BPSK with  $L$ -branch diversity and MRC is

$$\begin{aligned} P_b &= \int_0^\infty P_b(x) p_{\gamma_b^{\text{mrc}}}(x) dx \\ &= \int_0^\infty Q(\sqrt{2x}) \frac{1}{(L-1)! (\bar{\gamma}_c)^L} x^{L-1} e^{-x/\bar{\gamma}_c} \\ &= \left(\frac{1-\mu}{2}\right)^L \sum_{k=0}^{L-1} \binom{L-1+k}{k} \left(\frac{1+\mu}{2}\right)^k \end{aligned}$$

where

$$\mu = \sqrt{\frac{\bar{\gamma}_c}{1 + \bar{\gamma}_c}}$$

- The last step follows after considerable algebra.



*Bit error probability for BPSK with maximal ratio combining against the average branch bit energy-to-noise ratio.*



# Optimum Combining

- Maximum ratio combining (MRC) maximizes the output signal-to-noise ratio (SNR) and is the optimal combining method in a maximum likelihood sense for channels where the additive impairment is *additive white Gaussian noise*.
- When the additive channel impairment is dominated by *co-channel interference*, it is better to use **optimum combining** (OC) which is designed to maximize the output signal-to-interference-plus-noise ratio (SINR).
  - OC uses the spatial diversity not only to combat fading of the desired signal, as is the case with MRC, but also to reduce the relative power of the interfering signals at the receiver, such that the instantaneous SINR is maximized.
  - This is achieved by exploiting the correlation of the interference across the multiple receiver antenna elements.
  - By combining the signals that are received by multiple antennas, OC can suppress the interference and improve the output signal-to-interference-plus-noise ratio by several decibels in interference dominant environments.

# Received Signal, Interference and Noise

- Consider a situation where a desired signal is received in the presence of  $K$  co-channel interferers. The signal space dimensionality is assumed to be unity, i.e.,  $N = 1$ , so that the signal vectors are complex-valued scalars chosen from an appropriate constellation such as  $M$ -QAM.
- The received signal scalars at the  $L$  receiver antennas are equal to

$$\tilde{r}_k = g_{k,0}\tilde{s}_0 + \sum_{i=1}^K g_{k,i}\tilde{s}_i + \tilde{n}_k, \quad k = 1, \dots, L ,$$

where  $\tilde{s}_0$ ,  $\tilde{s}_i$  and  $\tilde{n}_k$  are the desired signal vector,  $i^{\text{th}}$  interfering signal vector, and noise vector, respectively, and  $K$  is the number of interferers.

- The  $L$  received signal scalars can be stacked in a column to yield the  $L \times 1$  received vector

$$\tilde{\mathbf{r}}_t = \mathbf{g}_0\tilde{s}_0 + \sum_{i=1}^K \mathbf{g}_i\tilde{s}_i + \tilde{\mathbf{n}} ,$$

where

$$\begin{aligned} \tilde{\mathbf{r}}_t &= (\tilde{r}_1, \tilde{r}_2, \dots, \tilde{r}_L)^T \\ \mathbf{g}_i &= (g_{i,1}, g_{i,2}, \dots, g_{i,L})^T \\ \tilde{\mathbf{n}} &= (\tilde{n}_1, \tilde{n}_2, \dots, \tilde{n}_L)^T . \end{aligned}$$

# Signal Correlations

- The  $L \times L$  received desired-signal-plus-interference-plus noise correlation matrix is given by

$$\Phi_{\tilde{\mathbf{r}}_t \tilde{\mathbf{r}}_t} = \frac{1}{2} \mathbb{E}_{\tilde{s}_0, \tilde{s}_i, \tilde{\mathbf{n}}} \left[ \left( \mathbf{g}_0 \tilde{s}_0 + \sum_{i=1}^K \mathbf{g}_i \tilde{s}_i + \tilde{\mathbf{n}} \right) \left( \mathbf{g}_0 \tilde{s}_0 + \sum_{i=1}^K \mathbf{g}_i \tilde{s}_i + \tilde{\mathbf{n}} \right)^H \right],$$

where  $(\cdot)^H$  denotes complex conjugate transpose.

- Likewise, the received interference-plus-noise correlation matrix is given by

$$\Phi_{\tilde{\mathbf{r}}_i \tilde{\mathbf{r}}_i} = \frac{1}{2} \mathbb{E}_{\tilde{s}_i, \tilde{\mathbf{n}}} \left[ \left( \sum_{i=1}^K \mathbf{g}_i \tilde{s}_i + \tilde{\mathbf{n}} \right) \left( \sum_{i=1}^K \mathbf{g}_i \tilde{s}_i + \tilde{\mathbf{n}} \right)^H \right].$$

- Note that the expectations are taken over a period that is much less than the channel coherence time, i.e., several modulated symbol durations.
- If the desired signal, interfering signal, and noise vectors are mutually uncorrelated,

$$\Phi_{\tilde{\mathbf{r}}_t \tilde{\mathbf{r}}_t} = \mathbf{g}_0 \mathbf{g}_0^H E_{\text{av}} + \sum_{i=1}^K \mathbf{g}_i \mathbf{g}_i^H E_{\text{av}}^i + N_o \mathbf{I}, \quad (2)$$

and

$$\Phi_{\tilde{\mathbf{r}}_i \tilde{\mathbf{r}}_i} = \sum_{i=1}^K \mathbf{g}_i \mathbf{g}_i^H E_{\text{av}}^i + N_o \mathbf{I},$$

respectively, where  $\mathbf{I}$  is the  $L \times L$  identity matrix and  $E_{\text{av}}^i$  is the average energy in the  $i^{\text{th}}$  interfering signal. It is important to note that the matrices  $\Phi_{\tilde{\mathbf{r}}_t \tilde{\mathbf{r}}_t}$  and  $\Phi_{\tilde{\mathbf{r}}_i \tilde{\mathbf{r}}_i}$  will vary at the channel fading rate.

## Optimum Combining and MMSE Solution

- The received signals scalars  $\tilde{r}_k, k = 1, 2, \dots, L$  are multiplied by controllable weights  $w_k$  and summed together, i.e., the combiner output is

$$\tilde{r} = \sum_{k=1}^L w_k \tilde{r}_k = \mathbf{w}^T \tilde{\mathbf{r}}_t ,$$

where  $\mathbf{w} = (w_1, w_2, \dots, w_L)^T$  is the weight vector.

- Several approaches can be taken to find the optimal weight vector  $\mathbf{w}$ . One approach is to minimize the mean square error

$$\begin{aligned} J &= \text{E} \left[ \|\tilde{r} - \tilde{s}_0\|^2 \right] \\ &= \text{E} \left[ \|\mathbf{w}^T \tilde{\mathbf{r}}_t - \tilde{s}_0\|^2 \right] \\ &= 2\mathbf{w}^T \Phi_{\tilde{\mathbf{r}}_t \tilde{\mathbf{r}}_t} \mathbf{w}^* - 2\text{Re} \{ \Phi_{\tilde{s}_0 \tilde{\mathbf{r}}_t} \mathbf{w}^* \} - 2E_{\text{av}} , \end{aligned}$$

where  $\Phi_{\tilde{\mathbf{r}}_t \tilde{\mathbf{r}}_t}$  is defined in (2) and

$$\Phi_{\tilde{s}_0 \tilde{\mathbf{r}}_t} = \text{E} [\tilde{s}_0 \tilde{\mathbf{r}}_t^H] = 2E_{\text{av}} \mathbf{g}_0^H .$$

- The weight vector that minimizes the mean square error can be obtained by setting the gradient  $\nabla_{\mathbf{w}} J$  to zero. This gives the minimum mean square error (MMSE) solution

$$\nabla_{\mathbf{w}} J = \left( \frac{\partial J}{\partial w_1}, \dots, \frac{\partial J}{\partial w_L} \right) = 2\mathbf{w}^T \Phi_{\tilde{\mathbf{r}}_t \tilde{\mathbf{r}}_t} - 2\Phi_{\tilde{s}_0 \tilde{\mathbf{r}}_t} = 0 .$$

- The solution is

$$\mathbf{w}_{\text{opt}} = \Phi_{\tilde{\mathbf{r}}_t \tilde{\mathbf{r}}_t}^{-1} \Phi_{\tilde{s}_0 \tilde{\mathbf{r}}_t}^T = 2E_{\text{av}} \Phi_{\tilde{\mathbf{r}}_t \tilde{\mathbf{r}}_t}^{-1} \mathbf{g}_0^* ,$$

where the fact that  $\Phi_{\tilde{s}_0 \tilde{\mathbf{r}}_t}^T = 2\mathbf{g}_0^* E_{\text{av}}$  was used.

# Optimum Combining and MMSE Solution

- Since  $\Phi_{\tilde{\mathbf{r}}_i \tilde{\mathbf{r}}_i} = \mathbf{g}_0 \mathbf{g}_0^H E_{\text{av}} + \Phi_{\tilde{\mathbf{r}}_i \tilde{\mathbf{r}}_i}$ , it follows that

$$\begin{aligned} \mathbf{w}_{\text{opt}} &= 2E_{\text{av}} (\Phi_{\tilde{\mathbf{r}}_i \tilde{\mathbf{r}}_i} + \mathbf{g}_0 \mathbf{g}_0^H E_{\text{av}})^{-1} \mathbf{g}_0^* \\ &= 2E_{\text{av}} (\Phi_{\tilde{\mathbf{r}}_i \tilde{\mathbf{r}}_i} + \mathbf{g}_0^* \mathbf{g}_0^T E_{\text{av}})^{-1} \mathbf{g}_0^* . \end{aligned} \quad (3)$$

- Next, a variation of the matrix inversion lemma is applied

$$(\mathbf{A} + \mathbf{u} \mathbf{v}^H)^{-1} = \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1} \mathbf{u} \mathbf{v}^H \mathbf{A}^{-1}}{1 + \mathbf{v}^H \mathbf{A}^{-1} \mathbf{u}}$$

to (3) resulting in

$$\begin{aligned} \mathbf{w}_{\text{opt}} &= 2E_{\text{av}} \left( \Phi_{\tilde{\mathbf{r}}_i \tilde{\mathbf{r}}_i}^{-1} - \frac{E_{\text{av}} \Phi_{\tilde{\mathbf{r}}_i \tilde{\mathbf{r}}_i}^{-1} \mathbf{g}_0^* \mathbf{g}_0^T \Phi_{\tilde{\mathbf{r}}_i \tilde{\mathbf{r}}_i}^{-1}}{1 + E_{\text{av}} \mathbf{g}_0^T \Phi_{\tilde{\mathbf{r}}_i \tilde{\mathbf{r}}_i}^{-1} \mathbf{g}_0^*} \right) \mathbf{g}_0^* \\ &= 2E_{\text{av}} \left( \frac{1}{1 + E_{\text{av}} \mathbf{g}_0^T \Phi_{\tilde{\mathbf{r}}_i \tilde{\mathbf{r}}_i}^{-1} \mathbf{g}_0^*} \right) \cdot \Phi_{\tilde{\mathbf{r}}_i \tilde{\mathbf{r}}_i}^{-1} \mathbf{g}_0^* \\ &= C \cdot \Phi_{\tilde{\mathbf{r}}_i \tilde{\mathbf{r}}_i}^{-1} \mathbf{g}_0^* , \end{aligned} \quad (4)$$

where  $C = 2E_{\text{av}} / (1 + E_{\text{av}} \mathbf{g}_0^T \Phi_{\tilde{\mathbf{r}}_i \tilde{\mathbf{r}}_i}^{-1} \mathbf{g}_0^*)$  is a scalar.

# Optimum Combining and Maximum SINR Solution

- Another criterion for optimizing the weight vector is to maximize the instantaneous signal-to-interference-plus-noise ratio (SINR) at the output of the combiner

$$\omega = \frac{\mathbf{w}^T \mathbf{g}_0 \mathbf{g}_0^H E_{\text{av}} \mathbf{w}^*}{\mathbf{w}^T \Phi_{\tilde{\mathbf{r}}_i \tilde{\mathbf{r}}_i}^{-1} \mathbf{w}^*} .$$

- Solving for the optimum weight vector gives

$$\mathbf{w}_{\text{opt}} = B \cdot \Phi_{\tilde{\mathbf{r}}_i \tilde{\mathbf{r}}_i}^{-1} \mathbf{g}_0^* ,$$

where  $B$  is an arbitrary constant. Hence, the *maximum* instantaneous output SINR is

$$\omega = E_{\text{av}} \mathbf{g}_0^H \Phi_{\tilde{\mathbf{r}}_i \tilde{\mathbf{r}}_i}^{-1} \mathbf{g}_0 .$$

- Note that the maximum instantaneous output SINR does not depend on the choice of the scalar  $B$ . Therefore, the MMSE weight vector in (4) also maximizes the instantaneous output SINR.
- When no interference is present,  $\Phi_{\tilde{\mathbf{r}}_i \tilde{\mathbf{r}}_i} = N_o \mathbf{I}$  and the optimal weight vector becomes

$$\mathbf{w}_{\text{opt}} = \frac{\mathbf{g}_0^*}{N_o} ,$$

so that the combiner output is

$$\tilde{r} = \sum_{k=1}^L \frac{g_{0,k}^*}{N_o} \tilde{r}_k .$$

– OC reduces to MRC when no interference is present.

# Performance of Optimum Combining

- To evaluate the performance of OC, several definitions are required as follows:

$$\Omega = \frac{\text{average received desired signal power per antenna}}{\text{average received noise plus interference power per antenna}}$$

$$\bar{\gamma}_c = \frac{\text{average received desired signal power per antenna}}{\text{average received noise power per antenna}} = \frac{E[|g_{0,k}|^2] E_{\text{av}}}{N_o}$$

$$\bar{\gamma}_i = \frac{\text{average received } i^{\text{th}} \text{ interferer power per antenna}}{\text{average received noise power per antenna}} = \frac{E[|g_{i,k}|^2] E_{\text{av}}^i}{N_o}$$

$$\omega_R = \frac{\text{instantaneous desired signal power at the array output}}{\text{average noise plus interference power at the array output}}$$

$$\omega = \frac{\text{instantaneous desired signal power at the array output}}{\text{instantaneous noise plus interference power at the array output}}$$

- In the above definitions, “average” refers to the average over the Rayleigh fading, while “instantaneous” refers to an average over a period that is much less than the channel coherence time, i.e., several modulated symbol durations so that the channel is essentially static.
- Note that

$$\Omega = \frac{\bar{\gamma}_c}{1 + \sum_{k=1}^K \bar{\gamma}_i} .$$

## Fading of the Desired Signal Only

- $\omega_R$  is equal to

$$\omega_R = E_{\text{av}} \mathbf{g}_0^H \Phi_{\tilde{\mathbf{r}}_i \tilde{\mathbf{r}}_i}^{-1} \mathbf{g}_0 ,$$

where, with a single interferer,

$$\Phi_{\tilde{\mathbf{r}}_i \tilde{\mathbf{r}}_i} = E_{\text{av}}^1 E[\mathbf{g}_1 \mathbf{g}_1^H] + N_o \mathbf{I} .$$

Note that the above expectation in is over the Rayleigh fading. The pdf of  $\omega_R$  is

$$p_{\omega_R}(x) = \frac{e^{-x/\bar{\gamma}_c} (x/\bar{\gamma}_c)^{L-1} (1 + L\bar{\gamma}_1)}{\bar{\gamma}_c (L-2)!} \int_0^1 e^{-((x/\bar{\gamma}_c)L\bar{\gamma}_1)t} (1-t)^{L-2} dt \quad (5)$$

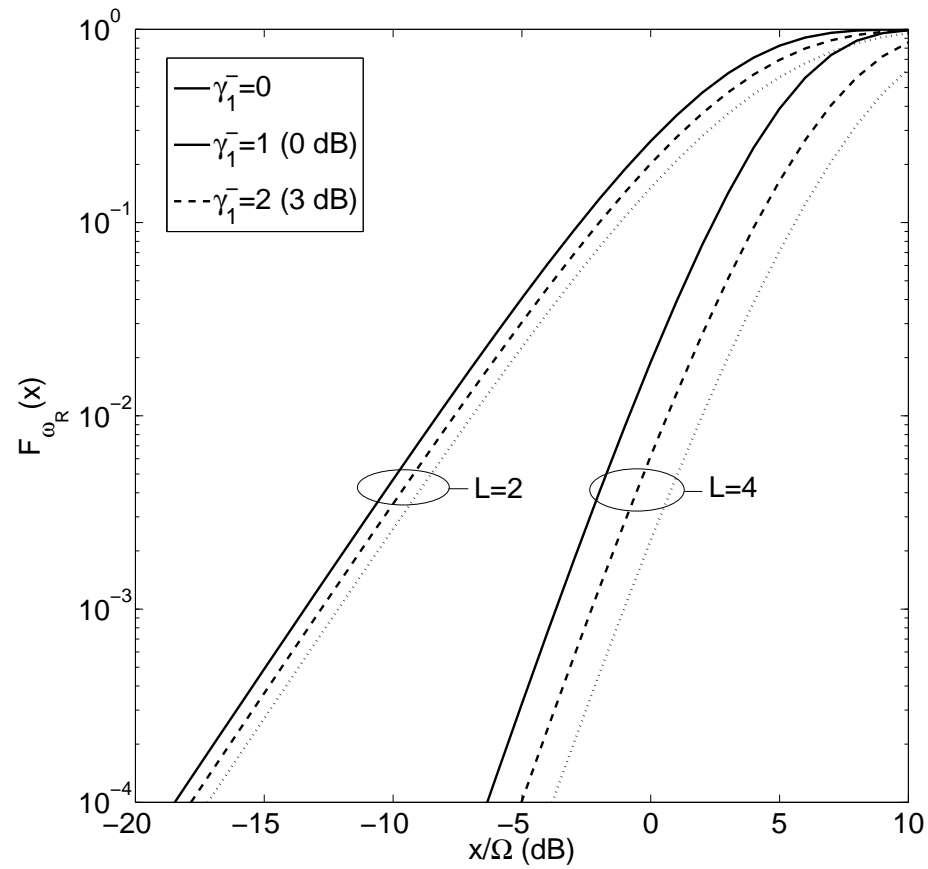
and the cdf of  $\omega_R$  is

$$F_{\omega_R}(x) = \int_0^{x/\bar{\gamma}_c} \frac{e^{-y} y^{L-1} (1 + L\bar{\gamma}_1)}{(L-2)!} \int_0^1 e^{-(yL\bar{\gamma}_1)t} (1-t)^{L-2} dt dy . \quad (6)$$

which are valid for  $L \geq 2$ .

- Note that  $\omega_R$  in (5) and (6) is normalized by  $\bar{\gamma}_c$ . Since  $\bar{\gamma}_c = (1 + \bar{\gamma}_1)\Omega$  for the case of a single interferer, it is apparent that  $\omega_R$  can be normalized by  $\Omega$  as well, i.e., replace  $x/\bar{\gamma}_c$  in the above pdf and cdf with  $x/(1 + \bar{\gamma}_1)\Omega$ . The normalization by  $\Omega$  allows for a straight forward comparison of OC and MRC.





*Cdf of  $\omega_R$  for optimal combining;  
 $\bar{\gamma}_c$  is the average branch symbol energy-to-noise ratio.*

## BER Performance

- The probability of bit error for coherently detected BPSK is given by

$$P_b = \int_0^\infty Q(\sqrt{2x}) p_{\omega_R}(x) dx$$

- Bogachev and Kieslev derived the bit error probability (for  $L \geq 2$ ) as

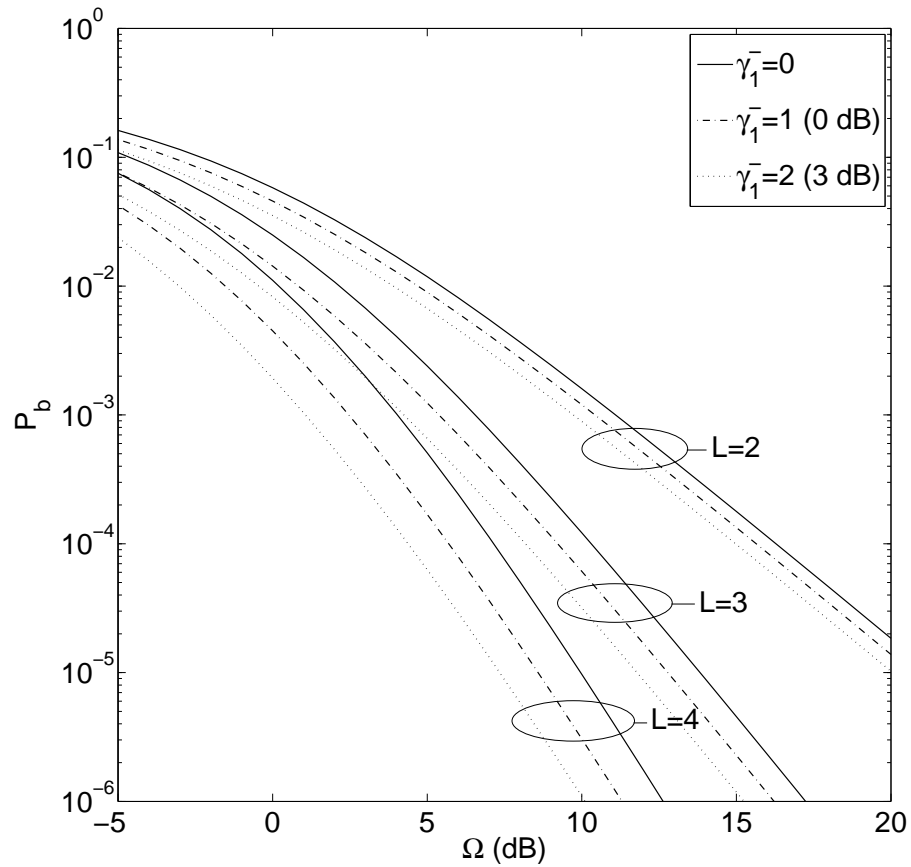
$$P_b = \frac{(-1)^{L-1}(1 + L\bar{\gamma}_1)}{2(L\bar{\gamma}_1)^{L-1}} \left\{ -\frac{L\bar{\gamma}_1}{1 + L\bar{\gamma}_1} + \sqrt{\frac{\bar{\gamma}_c}{1 + \bar{\gamma}_c}} - \frac{1}{1 + L\bar{\gamma}_1} \sqrt{\frac{\bar{\gamma}_c}{1 + L\bar{\gamma}_1 + \bar{\gamma}_c}} \right. \\ \left. - \sum_{k=1}^{L-2} (-L\bar{\gamma}_1)^k \left[ 1 - \sqrt{\frac{\bar{\gamma}_c}{1 + \bar{\gamma}_c}} \left( 1 + \sum_{i=1}^k \frac{(2i-1)!!}{i!(2 + 2\bar{\gamma}_c)^i} \right) \right] \right\}$$

where

$$(2i-1)!! = 1 \cdot 3 \cdot 5 \cdot \dots \cdot (2i-1) .$$

- Simon and Alouini have derived the following expression which is valid for  $L \geq 1$ :

$$P_b = \frac{1}{2} \left\{ 1 - \sqrt{\frac{\bar{\gamma}_c}{1 + \bar{\gamma}_c}} \sum_{k=0}^{L-2} \binom{2k}{k} \frac{1}{[4(1 + \bar{\gamma}_c)]^k} \left[ 1 - \left( -\frac{1}{L\bar{\gamma}_1} \right)^{L-1-k} \right] \right. \\ \left. - \sqrt{\frac{\bar{\gamma}_c}{1 + L\bar{\gamma}_1 + \bar{\gamma}_c}} \left( -\frac{1}{L\bar{\gamma}_1} \right)^{L-1} \right\}$$



*Bit error probability for coherent BPSK and optimal combining for various values of  $\bar{\gamma}_1$  and various number of receiver antenna elements,  $L$ .*

# Fading of the Desired and Interfering Signals

- The maximum instantaneous output SINR is equal to

$$\omega = E_{\text{av}} \mathbf{g}_0^H \Phi_{\tilde{\mathbf{r}}_i \tilde{\mathbf{r}}_i}^{-1} \mathbf{g}_0 ,$$

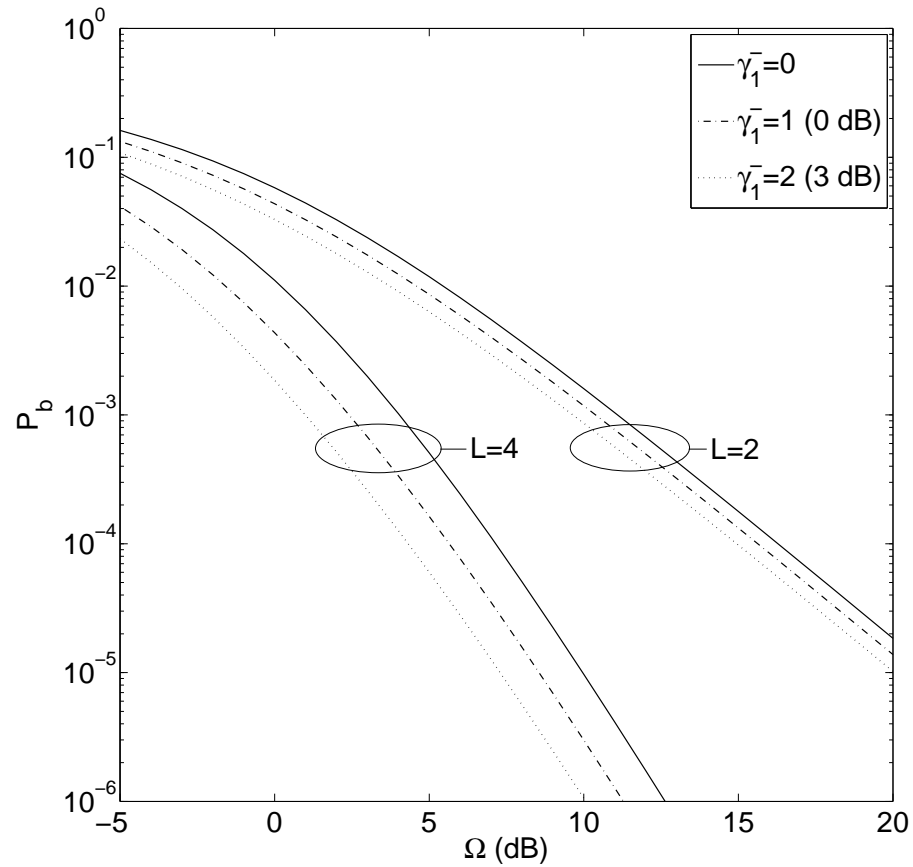
where, with a single interferer,

$$\Phi_{\tilde{\mathbf{r}}_i \tilde{\mathbf{r}}_i} = E_{\text{av}}^1 \mathbf{g}_1 \mathbf{g}_1^H + N_o \mathbf{I} .$$

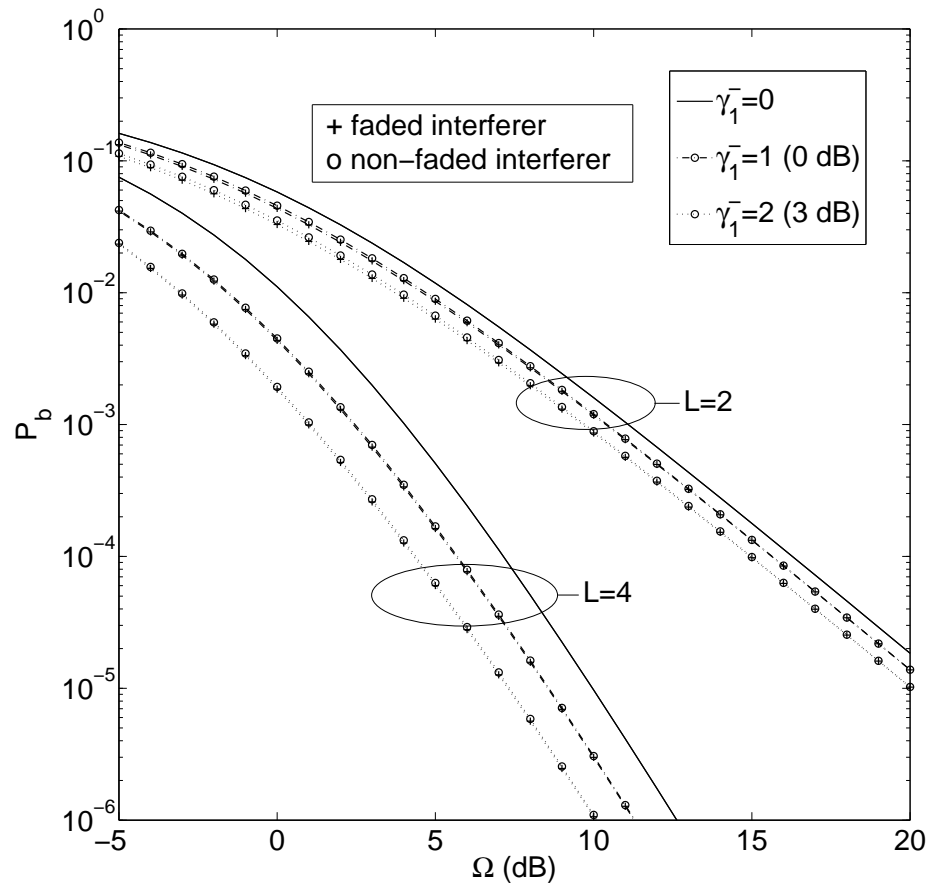
In this case, the matrix  $\Phi_{\tilde{\mathbf{r}}_i \tilde{\mathbf{r}}_i}$  varies at the fading rate.

- Using eigenvalue decomposition, the probability of bit error with coherent BPSK is

$$\begin{aligned} P_b &= \int_0^\infty P_{b|\gamma_1}(x) p_{\gamma_1}(x) dx \\ &= \frac{1}{2} \left[ 1 - \sqrt{\frac{\bar{\gamma}_c}{\bar{\gamma}_c + 1}} \sum_{k=0}^{L-2} \binom{2k}{k} \left( \frac{1}{4(\bar{\gamma}_c + 1)} \right)^k \right] \\ &\quad - \frac{1}{2\Gamma(L)(-\bar{\gamma}_1)^{L-1}} \left\{ \sqrt{\frac{\pi\bar{\gamma}_c}{\bar{\gamma}_1}} \exp\left(\frac{\bar{\gamma}_c + 1}{\bar{\gamma}_1}\right) \operatorname{erfc}\left(\sqrt{\frac{\bar{\gamma}_c + 1}{\bar{\gamma}_1}}\right) \right. \\ &\quad \left. - \sqrt{\frac{\bar{\gamma}_c}{\bar{\gamma}_c + 1}} \sum_{k=0}^{L-2} \frac{(2k)!}{k!} \left( \frac{-\bar{\gamma}_1}{4(\bar{\gamma}_c + 1)} \right)^k \right\} . \end{aligned}$$



*Bit error probability for coherent BPSK and optimal combining for various values of  $\bar{\gamma}_1$  and various number of receiver antenna elements,  $L$ .*



*Comparison of the bit error probability for coherent BPSK and optimal combining for a non-faded interferer and a faded interferer; the performance is almost identical.*