Solutions

Georgia Institute of Technology
School of Electrical and Computer Engineering

ECE6604 Personal & Mobile Communications

Final Exam

Fall 2017

Thursday December 14, 8:00am - 10:50am

- Attempt all questions.
- All questions are of equal value.
- Open book, open notes, exam.

1) Consider a linear *time-invariant* channel having the impulse response

$$g(t,\tau) = \delta(\tau) + 2\delta(\tau - \tau_1) + \delta(\tau - 2\tau_1) \ .$$

a) **(5 points)** Derive a closed-form expression for magnitude response of the channel $|T(f,t)|$ and sketch showing all important points.

b) **(2 points)** Repeat part a) for the phase response of the channel $\angle T(f,t)$.

c) **(3 points)** What is the mean delay and rms delay spread of the channel.

$$T(f,t) = 1 + 2e^{-j2\pi f\tau_1} + e^{-j4\pi f\tau_1}$$

$$= e^{-j2\pi f\tau_1}\left[e^{j2\pi f\tau_1} + 2 + e^{-j2\pi f\tau_1}\right]$$

$$= e^{-j2\pi f\tau_1}\left[2 + 2\cos(2\pi f\tau_1)\right]$$

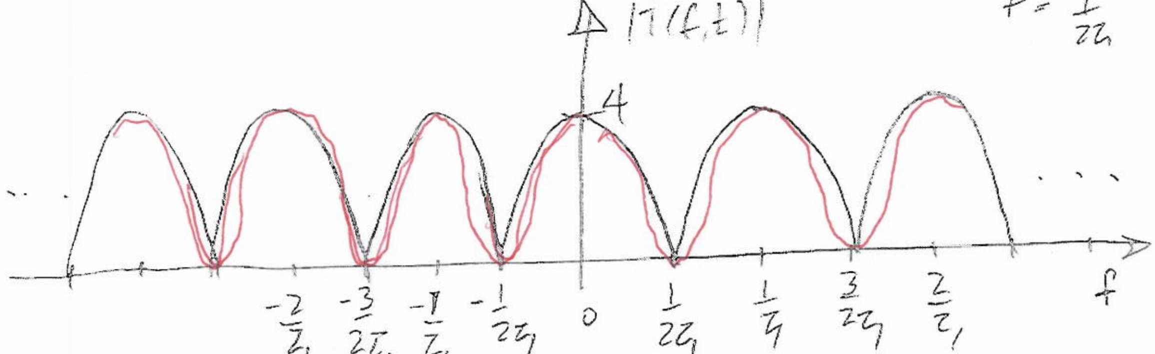
$$= 4e^{-j2\pi f\tau_1}\left[\frac{1+\cos(2\pi f\tau_1)}{2}\right]$$

$$= 4\cos^2(\pi f\tau_1)e^{-j2\pi f\tau_1}$$

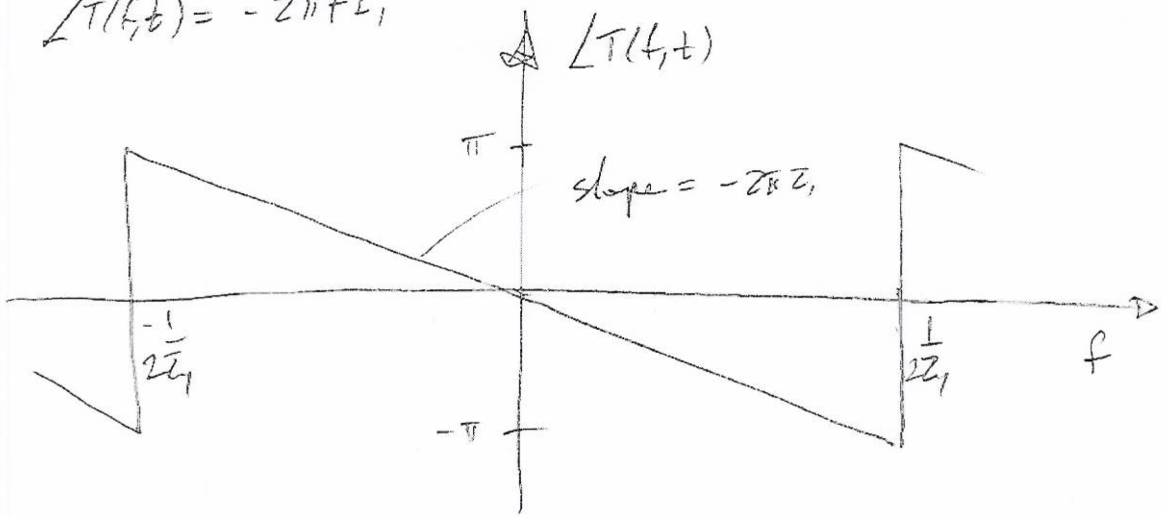
$$|T(f,t)| = 4\cos^2(\pi f\tau_1)$$

$$\pi f\tau_1 = \frac{\pi}{2}$$

$$f = \frac{1}{2\tau_1}$$



b) $\angle T(f,t) = -2\pi f \tau_1$



c)

Power delay profile

delay	power
0	1
τ_1	4
$2\tau_1$	1

$$\mu_\tau = \frac{\sum \tau_k P_k}{\sum P_k} = \frac{0\times 1 + \tau_1 \times 4 + 2\tau_1}{1+4+1}$$

$$= 6\tau_1/6 = \tau_1$$

$$\sigma_\tau^2 = \frac{\sum (\tau_k - \mu_\tau)^2 P_k}{\sum P_k}$$

$$= \frac{\tau_1^2 \times 1 + 0\times 4 + \tau_1^2 \times 1}{6} = \frac{2\tau_1^2}{6} = \frac{\tau_1^2}{3}$$

$$\sigma_\tau = \tau_1/\sqrt{3}$$

2) The power delay profile for a WSSUS channel is given by

$$\psi_g(\tau) = Ae^{-a\tau}u(\tau) + \frac{A}{2}e^{-a(\tau-\tau_d)}u(\tau-\tau_d)\ , \quad 0 \leq \tau \leq \infty$$

a) **3 marks:** Find the channel frequency correlation function.

b) **2 marks:** Find the total envelope power Ω_p.

c) **2 marks:** Find the mean delay μ_τ.

d) **3 marks:** Find the rms delay spread σ_τ.

a) $\phi_f(\Delta_f) \underset{F^{-1}}{\overset{F}{\longleftrightarrow}} \psi_g(\tau)$

$$\phi_f(\Delta_f) = \frac{A}{a+j2\pi\Delta_f}\left(1+\frac{1}{2}e^{-j2\pi\Delta_f\tau_d}\right)$$

b) $$\Omega_p = \int_0^\infty \psi_g(\tau)\,d\tau$$

$$= \int_0^\infty Ae^{-a\tau}d\tau + \int_{\tau_d}^\infty \frac{A}{2}e^{-a(\tau-\tau_d)}d\tau \qquad \hat{\tau} = \tau - \tau_d$$

$$= \frac{3A}{2}\int_0^\infty e^{-a\tau}d\tau$$

$$= -\frac{3A}{2}\cdot\frac{1}{a}e^{-a\tau}\Big|_0^\infty = \frac{3A}{2a}$$

Extra sheet

b) $\mu_\tau = \frac{1}{\Omega_p}\int_0^\infty \tau\, 4_g(\tau)\, d\tau$

$$= \frac{1}{\Omega_p}\int_0^\infty \tau A e^{-a\tau}\, d\tau + \frac{1}{\Omega_p}\int_{\tau_d}^\infty \tau \frac{A}{2} e^{-a(\tau-\tau_d)}\, d\tau$$

let $\hat{\tau} = \tau - \tau_d$ in second integral

$\tau = \hat{\tau} + \tau_d$

$$\mu_\tau = \frac{1}{\Omega_p}\int_0^\infty A\tau e^{-a\tau}\, d\tau + \frac{1}{\Omega_p}\int_0^\infty \frac{A}{2}(\hat{\tau}+\tau_d) e^{-a\hat{\tau}}\, d\hat{\tau}$$

$$= \frac{1}{\Omega_p}\frac{3A}{2}\int_0^\infty \tau e^{-a\tau}\, d\tau + \frac{1}{\Omega_p}\frac{A}{2}\tau_d\int_0^\infty e^{-a\tau}\, d\tau$$

$\int u\,dv = uv - \int v\,du \qquad u = \tau \qquad dv = e^{-a\tau}d\tau$

$\qquad du = d\tau \qquad v = -\frac{1}{a}e^{-a\tau}$

$$\int_0^\infty \tau e^{-a\tau}\, d\tau = -\frac{\tau}{a}e^{-a\tau}\Big|_0^\infty + \frac{1}{a}\int_0^\infty e^{-a\tau}\, d\tau$$

$$= -\frac{1}{a^2}e^{-a\tau}\Big|_0^\infty = \frac{1}{a^2}$$

$$\mu_\tau = \frac{1}{\Omega_p}\frac{3A}{2a^2} + \frac{1}{\Omega_p}\frac{A\tau_d}{2a}$$

$$= \frac{2a}{3A}\left(\frac{3A}{2a^2} + \frac{A\tau_d}{2a}\right) = \frac{1}{a} + \frac{\tau_d}{3}$$

$$\mu_\tau^2 = \frac{1}{a^2} + \frac{2\tau_d}{3a} + \frac{\tau_d^2}{9}$$

Extra sheet

$$\sigma_\tau^2 = \frac{\int_0^\infty \tau^2 \psi_g(\tau)\,d\tau}{\int_0^\infty \psi_g(\tau)\,d\tau} - \mu_\tau^2$$

$$= \frac{1}{\Omega_p}\int_0^\infty \tau^2 \psi_g(\tau)\,d\tau - \mu_\tau^2$$

$$\frac{1}{\Omega_p}\int_0^\infty \tau^2 \psi_g(\tau)\,d\tau = \frac{1}{\Omega_p}\int_0^\infty A\tau^2 e^{-a\tau}\,d\tau + \frac{1}{\Omega_p}\int_{\tau_d}^\infty \frac{A}{2}\tau^2 e^{-a(\tau-\tau_d)}\,d\tau$$

$$= \frac{1}{\Omega_p}A\int_0^\infty \tau^2 e^{-a\tau}\,d\tau + \frac{1}{\Omega_p}\frac{A}{2}\int_0^\infty (\hat{\tau}+\tau_d)^2 e^{-a\hat{\tau}}\,d\hat{\tau}$$

$$= \frac{1}{\Omega_p}A\int_0^\infty \tau^2 e^{-a\tau}\,d\tau + \frac{1}{\Omega_p}\frac{A}{2}\int \tau^2 e^{-a\tau}\,d\tau$$

$$+ \frac{1}{\Omega_p}A\tau_d\int_0^\infty \tau e^{-a\tau}\,d\tau + \frac{1}{\Omega_p}\frac{A}{2}\tau_d^2\int_0^\infty e^{-a\tau}\,d\tau$$

$$= \frac{1}{\Omega_p}\frac{3A}{2}\cdot\frac{2}{a^3} + \frac{1}{\Omega_p}A\tau_d\cdot\frac{1}{a^2} + \frac{1}{\Omega_p}\frac{A}{2}\frac{\tau_d^2}{a}$$

$$= \frac{2\cancel{a}}{3\cancel{A}}\left(\frac{3A}{a^{\cancel{3}\,2}} + \frac{\cancel{A}\tau_d}{a^{\cancel{2}\,1}} + \frac{\cancel{A}\tau_d^2}{2\cancel{a}}\right)$$

$$= \frac{2}{a^2} + \frac{2\tau_d}{3a} + \frac{\tau_d^2}{3}$$

$$\sigma_\tau^2 = \frac{1}{a^2} + \frac{2\tau_d^2}{9}$$

$$\sigma_\tau = \sqrt{\frac{1}{a^2} + \frac{2\tau_d^2}{9}}$$

3) Consider a cellular system that uses a 3-cell hexagonal reuse cluster. The base stations employ 120° wide-beam directional antennas and they all have the same antenna height and transmit with the same power level. The cell radii are assumed to be 5 km.

The propagation path loss follows the model

$$\mu_{\Omega_p \ (\mathrm{dBm})}(d) = \mu_{\Omega_p \ (\mathrm{dBm})}(d_o) - 10\beta \log_{10}(d/d_o) \quad (\mathrm{dBm})$$

where $\beta = 3.5$, and $\mu_{\Omega_p}(d_o) = 1$ microwatt at $d_o = 1$ km. Assume that each link experiences independent log-normal shadowing with a shadow standard deviation $\sigma_\Omega = 6$ dB.

Consider the forward (base-to-mobile) channel.

a) **2 marks:** Using the attached hex paper depict the worst-case co-channel interference situation on the forward channel, accounting only for the first tier of co-channel interferers.

b) **6 marks:** Using the Fenton-Wilkinson method, determine the probability density function of the total interfering power in decibel units, $I_{(\mathrm{dBm})}$, again accounting only for the first tier of co-channel interferers.

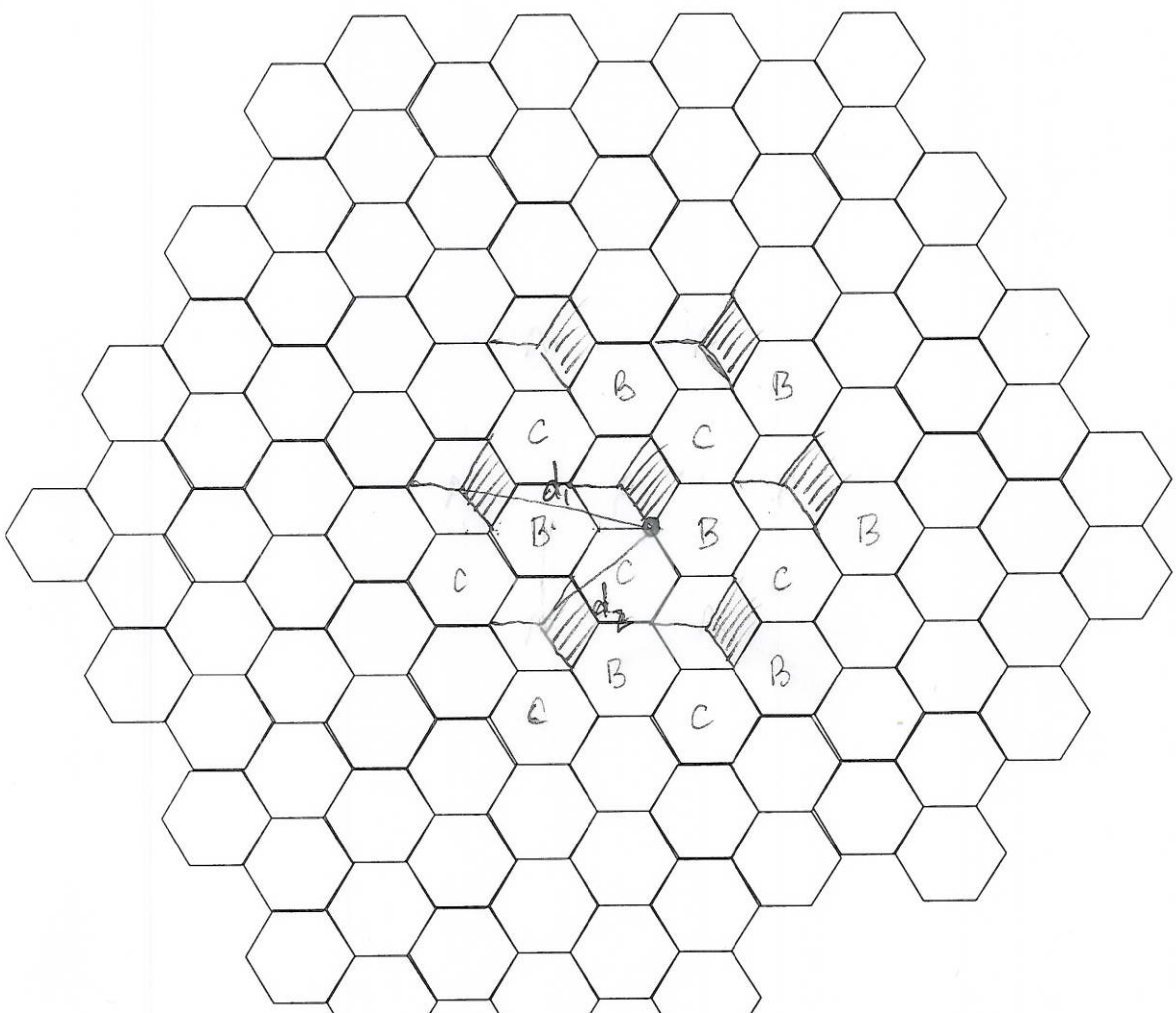
c) **2 marks:** Using the result in part b), determine the probability density function of the carrier-to-interference ratio in decibel units, $(C/I)_{(\mathrm{dB})}$.

$$d_1^2 = \left(\frac{\sqrt{3}}{2}R\right)^2 + \left(\frac{7}{2}R\right)^2 = \frac{3}{4}R^2 + \frac{49}{4}R^2 = 13R^2$$

$$d_1 = \sqrt{13}\,R$$

$$d_2^2 = \left(2\frac{\sqrt{3}}{2}R\right)^2 + (2R)^2 = 3R^2 + 4R^2 = 7R^2$$

$$d_2 = \sqrt{7}\,R$$



Extra sheet

b) Two co-channel interferers at distances $d_1 = \sqrt{13}R$ and $d_2 = \sqrt{7}R$

$$d_1 = 5\sqrt{13}\text{ km} \quad d_2 = 5\sqrt{7}\text{ km}$$

$$\mu_{\Omega_1} = -30 - 10(3.5)\log_{10}(5\sqrt{13}) = -73.958 \text{ dBm}$$

$$\mu_{\Omega_2} = -30 - 10(3.5)\log_{10}(5\sqrt{7}) = -69.253 \text{ dBm}$$

$$\mu_{\hat{\Omega}_k} = \xi\mu_{\Omega_k} \qquad \xi = \frac{\ln 10}{10} = 0.23026$$

$$\mu_{\hat{\Omega}_1} = (.23026)(-73.958) = -17.030$$

$$\mu_{\hat{\Omega}_2} = (.23026)(-69.253) = -15.946$$

$$\sigma_{\hat{\Omega}}^2 = \xi^2\sigma_{\Omega}^2 = (.23026)^2(36) = 1.909$$

$$\sigma_{\hat{Z}}^2 = \ln\left((e^{1.909}-1)\frac{e^{-34.060} + e^{-31.892}}{(e^{-17.030} + e^{-15.946})^2} + 1\right)$$

$$= \ln(5.746 \times 0.6223 + 1)$$

$$= \ln(4.5757) = 1.5208$$

$$\sigma_Z^2 = \sigma_{\hat{Z}}^2/\xi^2 = 28.683$$

$$\sigma_Z = 5.356$$

Extra sheet

$$\mu_{\hat{Z}} = \frac{\sigma_{\hat{\Omega}}^2 - \sigma_{\hat{Z}}^2}{2} + \ln\left(e^{\mu_{\hat{\Omega}_1}} + e^{\mu_{\hat{\Omega}_2}}\right)$$

$$= \frac{1.909 - 1.5208}{2} + \ln\left(e^{-17.030} + e^{-15.946}\right)$$

$$= 0.1941 - 15.655$$

$$= -15.461$$

$$\mu_Z = \mu_{\hat{Z}}/\xi = -67.144$$

$$\therefore I = I_1 + I_2 \sim N(-67.144,\ 28.683)$$

c) $\mu_{\Omega d} = -30 - 10(3.5)\log_{10}(5) = -54.464$ dBm

$$\sigma_d^2 = \sigma_\Omega^2 = 36$$

$$\Lambda_{dB} = C_{(dBm)} - I_{(dBm)}$$

$$C(dBm) \sim N(-54.464,\ 36)$$

$$\therefore \Lambda_{dB} \sim N(12.68,\ 64.683)$$

4) Consider coherent BPSK signaling on an AWGN channel where the channel gain, α, has the following probability density function

$$p_\alpha(x) = 0.2\delta(x) + 0.5\delta(x-1) + 0.3\delta(x-2) \ .$$

Suppose that two-branch antenna diversity is used with *selective combining*. Assume that the diversity branches experience independent fading.

a) 5 **marks:** Derive an expression for the probability density function of the bit-energy-to-noise ratio at the output of the selective combiner, γ_s^{b}.

b) 2 **marks:** Express the probability density function obtained in part a) as a function of the average received bit energy-to-noise ratio per diversity branch, $\bar{\gamma}_c$.

c) 2 **marks:** Derive the probability of bit error a function of the average received bit energy-to-noise ratio per diversity branch $\bar{\gamma}_c$.

d) 1 **marks:** What value does the probability of bit error approach as $\bar{\gamma}_c$ tends to infinity?

First derive the pdf of the effective value of α, called α^c, at the output of the selective combiner. For two branch selective diversity

$$\alpha^c = \max(\alpha_1, \alpha_2)$$

First derive the cdf of α^c

$$F_{\alpha^c}(x) = P(\alpha_1 \le x, \alpha_2 \le x)$$
$$= P(\alpha_1 \le x)\,P(\alpha_2 \le x)$$

$$F_{\alpha^c}(x) = 0, \quad x < 0$$

$$F_{\alpha^c}(0) = P(\alpha_1 \le 0)\,P(\alpha_2 \le 0)$$
$$= 0.2 \times 0.2 = 0.04$$

$$F_{\alpha^c}(x) = 0.04 \quad 0 \le \alpha^c < 1$$

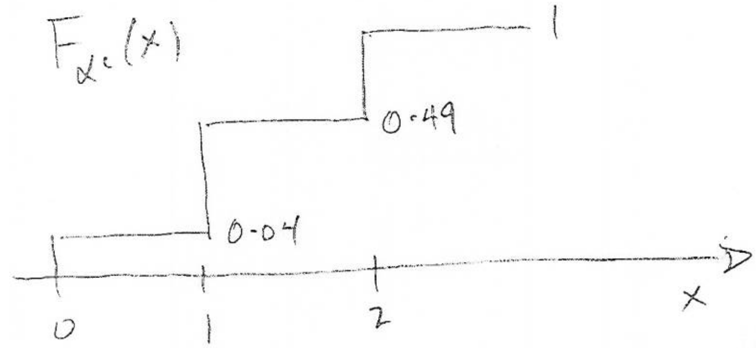
$$F_{\alpha^c}(1) = P(\alpha_1 < 1)\,P(\alpha_2 < 1)$$
$$= 0.7 \times 0.7 = 0.49$$

$$F_{\alpha^c}(x) = 0.49. \quad 1 \le x < 2$$

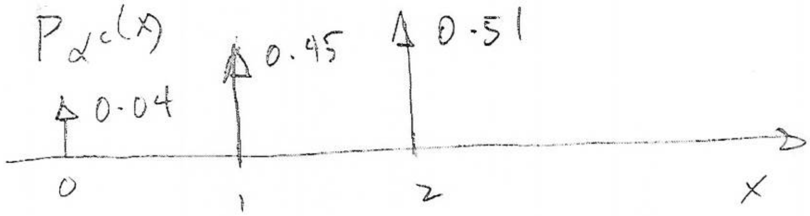
$$F_{\alpha^c}(2) = P(\alpha_1 < 2)\,P(\alpha_2 < 2)$$
$$= 1 \times 1 = 1$$

$$F_{\alpha^c}(x) = 1 \quad x \geq 2$$

4/



Taking derivative gives $P_{\alpha_c}(x)$ below



Then

$$\gamma_a = \alpha^2 \frac{E_b}{N_0}$$

$$\bar{\gamma}_c = 0\cdot2 \times 0 \times \frac{E_b}{N_0} + 0\cdot5 \times 1 \times \frac{E_b}{N_0} + 0\cdot3 \times 4 \frac{E_b}{N_0}$$

$$= 1\cdot7 \frac{E_b}{N_0} \Rightarrow \frac{E_b}{N_0} = \frac{\bar{\gamma}_c}{1\cdot7}$$

Hence, $\quad P_b(\gamma_b) = Q\left(\sqrt{2\gamma_b}\right), \quad \gamma_b = \alpha_c^2 \frac{E_b}{N_0}$

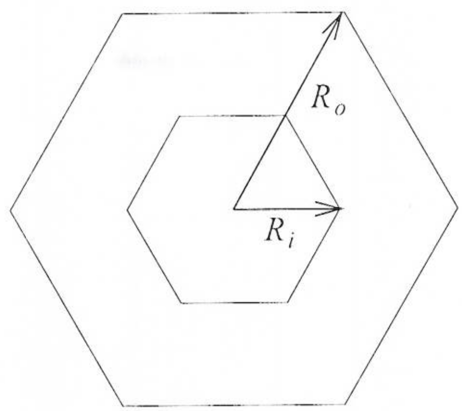
$$P_b = 0\cdot04\, \underset{0.5}{Q(0)} + 0\cdot45\, Q\left(\sqrt{\frac{2E_b}{N_0}}\right) + 0\cdot51\, Q\left(\sqrt{2\cdot 4\frac{E_b}{N_0}}\right)$$

$$= 0\cdot02 + 0\cdot45\, Q\left(\sqrt{1.176\,\bar{\gamma}_c}\right) + 0\cdot51\, Q\left(\sqrt{4.706\,\bar{\gamma}_c}\right)$$

As $\bar{\gamma}_c \to \infty$

$$P_b \to 0\cdot02$$

5) One method for improving the capacity of a cellular system is to employ a *two-channel bandwidth* scheme, where each hexagonal cell is divided into two concentric hexagons as shown below. The inner hexagon is serviced by half-rate channels, while the outer hexagon is serviced by full-rate channels. When a mobile station crosses the boundary between the inner and outer portions of a cell a handoff occurs.



It is known that the full-rate channels require $C/I = 7$ dB to maintain an acceptable radio link quality, while the half-rate channels require $C/I = 10$ dB.

Assume a $\beta = 4$ path loss exponent and suppose that the effects of envelope fading and shadowing can be ignored. Consider the reverse link and suppose that there are 6 co-channel interferers at distance D from the serving base station. It follows that the received C/I when a mobile station is located at distance d from its serving base station is $C/I = (D/d)^4/6$.

a) **2 marks:** Find the required value of D/R_o so that the worst case $C/I = 7$ dB in the outer cell, where R_o is the radii of the outer cell.

b) **2 marks:** Find the required value of D/R_i so that the worst case $C/I = 10$ dB in the inner cell, where R_i is the radii of the inner cell.

c) **3 marks:** Use the values of D/R_i and D/R_o to determine the ratio of the inner and outer cell areas, A_i/A_o. Use the exact area of a hexagon in terms of its radius.

d) **3 marks:** Let N_i and N_o be the number of channels that are allocated to the inner and outer areas of each cell, and assume that the channels are assigned such that $N_i/N_o = A_i/A_o$. Determine the increase in cell capacity (as measured in channels per cell) over a conventional *one-channel bandwidth* system that uses only full-rate channels.

a) $$\frac{C}{I} = \frac{(D/R_o)^4}{6} = 7\,dB$$

$$D/R_o = (6 \times 5.01)^{1/4} = 2.34$$

b) $$\frac{(D/R_i)^4}{6} = 10\,dB$$

$$D/R_i = (6 \times 10)^{1/4} = 2.78$$

c) $$\frac{D/R_i}{D/R_o} = \frac{R_o}{R_i} = \frac{2.78}{2.34} = 1.19$$

$$A_o = \frac{3\sqrt{3}}{2} R_o^2 - \frac{3\sqrt{3}}{2} R_i^2$$

$$\frac{A_i}{A_o} = \frac{\frac{3\sqrt{3}}{2} R_i^2}{\frac{3\sqrt{3}}{2} R_o^2 - \frac{3\sqrt{3}}{2} R_i^2}$$

$$= \frac{R_i^2}{R_o^2 - R_i^2}$$

$$= \frac{1}{(R_o/R_i)^2 - 1}$$

$$= \frac{1}{(1.19)^2 - 1} = 2.4033$$

d) Suppose there are N_T full rate channels to start with.

Let $\quad N_i = 2\beta N_T \quad$ half rate

$\quad\quad N_0 = (1-\beta) N_T \quad$ full rate

Want $\quad \frac{N_i}{N_0} = \frac{A_i}{A_0} = \frac{2\beta}{1-\beta} = 2.4033$

$\Rightarrow 2\beta = 2.4033 - 2.4033\beta$

$\Rightarrow \beta = 0.54579$

$\therefore \quad N_0 = (1-\beta) N_T = 0.4542 N_T$

$\quad\quad N_i = 2\beta N_T = 1.09158 N_T$

$N_0 + N_i = 1.5458 N_T$

$\Rightarrow$ increase by factor 1.5458 or 54.58 %