ECE 6604 Assignment #2
Solutions

2.6 a) The bandpass Doppler power spectrum is Eqs (2.33) and (2.34) in text

$$S_{rr}(f) = \frac{\Omega_p/4}{\sqrt{f_m^2-(f-f_c)^2}}\left\{p(\theta)+p(-\theta)\right\} \qquad \theta = \cos^{-1}\left(\frac{f-f_c}{f_m}\right)$$

$$= \frac{\Omega_p}{8\pi I_0(k) f_m \sqrt{1-\left(\frac{f-f_c}{f_m}\right)^2}}\left\{e^{k\cos(\mu-\theta)} + e^{k\cos(\mu+\theta)}\right\}$$

But

$$\{\cdot\} = e^{k\cos\mu\cos\theta + k\sin\mu\sin\theta} + e^{k\cos\mu\cos\theta - k\sin\mu\sin\theta}$$

$$= 2e^{k\cos\mu\cos\theta}\left(\frac{e^{k\sin\mu\sin\theta}+e^{-k\sin\mu\sin\theta}}{2}\right)$$

$$= 2e^{k\cos\mu\cos\theta}\cosh(k\sin\mu\sin\theta)$$

$$= 2e^{k(\cos\mu)(f-f_c)/f_m}\cosh\left(k(\sin\mu)\sqrt{1-\left(\frac{f-f_c}{f_m}\right)^2}\right)$$

Hence

$$S_{rr}(f) = \frac{\Omega_p}{4\pi I_0(k) f_m\sqrt{1-\left(\frac{f-f_c}{f_m}\right)^2}} \cdot e^{k(\cos\mu)\left(\frac{f-f_c}{f_m}\right)}\cosh\left(k(\sin\mu)\sqrt{1-\left(\frac{f-f_c}{f_m}\right)^2}\right)$$

$$|f-f_c| \le f_m$$

Then,

$$S_{gg}(f) = \frac{\Omega_p}{2\pi I_0(k) f_m\sqrt{1-(f/f_m)^2}}\, e^{k(\cos\mu)\frac{f}{f_m}}\cosh\left(k(\sin\mu)\sqrt{1-(f/f_m)^2}\right)$$

$$|f| \le f_m$$

b) Need to show $\phi_{g_I g_Q}(\tau) = 0$

$$\phi_{g_I g_Q}(\tau) = \frac{\Omega_p}{2} E_\theta\left[\sin(2\pi f_m \tau \cos\theta)\right]$$

$$= \frac{\Omega_p}{2} \int_{-\pi}^{\pi} \sin(2\pi f_m \tau \cos\theta) \frac{1}{2\pi I_0(k)} e^{k\cos(\theta-\mu)} d\theta$$

$$= \frac{\Omega_p}{8\pi j I_0(k)}\left[\int_{-\pi}^{\pi} e^{j2\pi f_m \tau \cos\theta + k\cos(\theta-\mu)} d\theta - \int_{-\pi}^{\pi} e^{-j2\pi f_m \tau \cos\theta + k\cos(\theta-\mu)} d\theta\right]$$

$$= \frac{\Omega_p}{8\pi j I_0(k)}\left[\int_{-\pi}^{\pi} e^{(j2\pi f_m \tau + k\cos\mu)\cos\theta + k\sin\mu\sin\theta} d\theta - \int_{-\pi}^{\pi} e^{(-j2\pi f_m \tau + k\cos\mu)\cos\theta + k\sin\mu\sin\theta} d\theta\right]$$

$$= \frac{\Omega_p}{8\pi j I_0(k)}\left[\int_{0}^{2\pi} e^{(-j2\pi f_m \tau - k\cos\mu)\cos\theta - k\sin\mu\sin\theta} d\theta - \int_{0}^{2\pi} e^{(j2\pi f_m \tau - k\cos\mu)\cos\theta - k\sin\mu\sin\theta} d\theta\right]$$

$$= \frac{\Omega_p}{8\pi j I_0(k)}\left[2\pi I_0\left(\sqrt{(k\cos\mu + j2\pi f_m \tau)^2 + (k\sin\mu)^2}\right) - 2\pi I_0\left(\sqrt{(k\cos\mu - j2\pi f_m \tau)^2 + (k\sin\mu)^2}\right)\right]$$

$$= \frac{\Omega_p}{j4 I_0(k)}\left[I_0\left(\sqrt{k^2 - 4\pi^2(f_m\tau)^2 + j4\pi k f_m \tau \cos\mu}\right) - I_0\left(\sqrt{k^2 - 4\pi^2(f_m\tau)^2 - j4\pi k f_m \tau\cos\mu}\right)\right]$$

To get uncorrelated $g_I(t)$ and $g_Q(t)$ we need to have

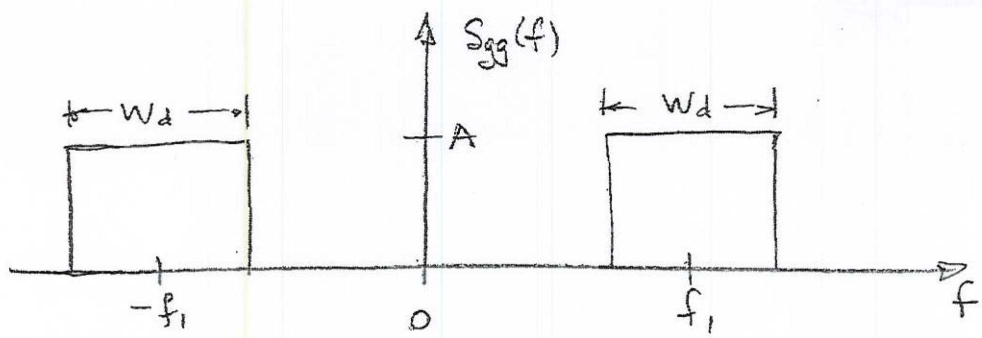
$$4\pi k f_m \tau \cos\mu = 0$$

Hence, we must have

$k = 0$ (isotropic scattering)

or $\mu = \pm \frac{\pi}{2}$ (mean angle of arrival is perpendicular to the direction of motion)

2) 2.11 a)



b) From Fourier transform tables:

$$A\,\mathrm{rect}\left(\frac{f}{W_d}\right) \longleftrightarrow A W_d\, S_a(\pi W_d \tau)$$

Using the modulation property of Fourier transform, and the above,

$$\phi_{gg}(\tau) = 2AW_d\, S_a(\pi W_d \tau) \cos(2\pi f_1 \tau)$$

c) $g(t)$ and $g(t+\tau)$ are uncorrelated when $\phi_{gg}(\tau) = 0$

$Sa(\pi W_d \tau) = 0$ when

$W_d \tau = k$, k an integer, $k \neq 0$

$$\tau = \frac{k}{W_d}$$

$\cos(2\pi f_1 \tau) = 0$ when

$2\pi f_1 \tau = l\frac{\pi}{2}$, l an odd integer

$$\tau = \frac{l}{4f_1}$$

Hence, $\phi_{gg}(\tau) = 0$ when

$$\tau = \begin{cases} \dfrac{k}{W_d}, & k \text{ an integer}, k \neq 0 \\ \dfrac{l}{4f_1}, & l \text{ an odd integer.} \end{cases}$$

3) 2.17 For $f_c = 6$ GHz, and $v = 80$ km/h

$$f_m = \frac{v}{c} \cdot f_c = \frac{80\times10^3/(60\times60)}{3\times10^8}(6\times10^9)$$

$$= 444.\dot{4} \text{ Hz}$$

a) $$L_R = \sqrt{2\pi} f_m \rho e^{-\rho^2} \quad \text{where } \rho = \frac{R}{R_{rms}} = 1$$

$$= \sqrt{2\pi} \times 444.\dot{4} \cdot 1 \cdot e^{-1}$$

$$= 409.8 \text{ crossings/sec on avg.}$$

In 5 s. there are

$$409.8 \times 5 = 2049.19 \text{ crossings on average}$$

b) $$\bar{t} = \frac{e^{\rho^2} - 1}{\rho f_m \sqrt{2\pi}} = \frac{e-1}{444.4\sqrt{2\pi}}$$

$$= 1.542 \text{ ms}$$

$$p(\alpha, \dot{\alpha}) = \sqrt{\frac{1}{2\pi b_2}}\, e^{\overline{2b_2}} \cdot \frac{\alpha}{b_0} e \qquad = p(\alpha) \cdot p(\dot{\alpha})$$

c) For $\rho = -20$ dB $\quad \rho = 10^{\rho_{dB}/20} = 0.1$

$$\bar{t} = \frac{e^{\rho^2} - 1}{\rho f_m \sqrt{2\pi}} = \frac{e^{(0.1)^2} - 1}{(0.1)\, 444.4\sqrt{2\pi}}$$

$$= 90.21\ \mu s.$$

4) 2.20 a) $\rho = R/\sqrt{\Omega_p} = 1$

$L_R = \sqrt{2\pi} f_m \rho e^{-\rho^2}$

$= \sqrt{2\pi} \times 20 \times e^{-1}$

$= \boxed{18.44 \text{ crossings/second}}$

$f_m = \frac{v}{\lambda_c} \quad c = f_c \lambda_c$

$\Rightarrow v = f_m \lambda_c = \frac{f_m c}{f_c} = \frac{20 \times 3 \times 10^8}{900 \times 10^6}$

$= \boxed{6.67 \text{ m/s}}$

b) $f_m = \frac{v}{\lambda_c} = \frac{v f_c}{c} = \frac{24 \times 10^3 \times 900 \times 10^6}{3600 \times 3 \times 10^8}$

$= .20 \text{ Hz}$

$\bar{t} = \frac{e^{\rho^2} - 1}{\rho f_m \sqrt{2\pi}} = \boxed{6.125 \text{ ms}}$ @ $\rho = .294$

$L_R = \sqrt{2\pi} f_m \rho e^{-\rho^2} = \boxed{13.52 \text{ crossings/second}}$

@ $\rho = .294$

5) 2.21 Assuming the received signal is Rayleigh faded, the average fade duration (equation 2.119 in the text) and the maximum Doppler frequency are given by, respectively,

$$\bar{t} = \frac{e^{\rho^2}-1}{\rho f_m \sqrt{2\pi}} \quad \Rightarrow \quad f_m = \frac{v}{\lambda_c} = \frac{e^{\rho^2}-1}{\bar{t}\rho\sqrt{2\pi}}$$

For $f_c = 900$ MHz, $\lambda_c = c/f_c = \frac{3\cdot 10^8}{9\cdot 10^8} = 0.33$ m. Then, for $\rho = -10$ dB ($\rho = 0.316$) and $\bar{t} = 1$ ms, the vehicle speed $v = 442$ m/s. Hence, mobile goes 442 m in 10 s.

At rms threshold level $\rho = R/R_{\text{rms}} = 1$. The level crossing rate is given by (equation 2.111 in the text)

$$L_R = \sqrt{2\pi} f_m \rho e^{-\rho^2}$$

Then, the level crossing rate at rms level is $L_{R_{\text{rms}}} = 95.15$. Hence the received signal experiences $10 \cdot L_{R_{\text{rms}}} \approx 952$ fades below the rms threshold level during 10 s interval.

6) 2.24

$$\phi_T(f,m;t,s) = \phi_T(m+\Delta f, m; t,s) = \phi_T(\Delta f; t,s)$$

$$\begin{aligned}
\phi_g(t,s;\tau,\eta) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_T(f,m;t,s)\, e^{j2\pi(\tau f-\eta m)}\, df\, dm \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_T(\Delta f;t,s)\, e^{j2\pi(\tau m-\eta m+\tau\Delta f)}\, dm\, d\Delta f \\
&= \int_{-\infty}^{\infty} e^{-j2\pi(\eta-\tau)m}\, dm \int_{-\infty}^{\infty} \phi_T(\Delta f;t,s)\, e^{j2\pi\tau\Delta f}\, d\Delta f \\
&= \delta(\eta-\tau)\, \psi_g(t,s;\tau)
\end{aligned}$$

$$\phi_H(f,m;\nu,\mu) = \phi_H(m+\Delta f, m;\nu,\mu) = \phi_H(\Delta f;\nu,\mu)$$

$$\begin{aligned}
\phi_S(\tau,\eta;\nu,\mu) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_H(f,m;\nu,\mu)\, e^{j2\pi(\tau f-\eta m)}\, df\, dm \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_H(\Delta f;\nu,\mu)\, e^{j2\pi(\tau m-\eta m+\tau\Delta f)}\, dm\, d\Delta f \\
&= \int_{-\infty}^{\infty} e^{-j2\pi(\eta-\tau)m}\, dm \int_{-\infty}^{\infty} \phi_H(\Delta f;\nu,\mu)\, e^{j2\pi\tau\Delta f}\, d\Delta f \\
&= \delta(\eta-\tau)\, \psi_S(\tau;\nu,\mu)
\end{aligned}$$

Singularity with respect to the time variable implies that multipath components arriving at different delays and, hence, distance have uncorrelated fading.