ECE6604 Assignment #5

Solutions

4.9.

$$PAPR = \frac{\max_m |X_{n,m}|^2}{\frac{1}{N}\sum_{m=0}^{N-1} |X_{n,m}|^2}$$

where

$$X_{n,m} = \sum_{k=0}^{N-1} x_{n,k} e^{j\frac{2\pi km}{N}}$$

Also

$$x_{n,k} = \frac{1}{N}\sum_{m=0}^{N-1} X_{n,m} e^{-j\frac{2\pi km}{N}}$$

From Parseval's theorem

$$\frac{1}{N}\sum_{m=0}^{N-1} |X_{n,m}|^2 = \sum_{k=0}^{N-1} |x_{n,k}|^2 = N$$

since $x_{n,k} \in \{-1,+1\}$

From triangle inequality

$$|X_{n,m}| = \left|\sum_{k=0}^{N-1} x_{n,k} e^{j\frac{2\pi kn}{N}}\right|$$

$$\leq \sum_{k=0}^{N-1} |x_{n,k}| = N \text{ since } \left|e^{j\frac{2\pi kn}{N}}\right| = 1$$

$\therefore$ $PAPR \leq \frac{N^2}{N} = N$

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2/ 4.12 Since the PAPR is determined through simulations, your results may differ slightly from the answers below.

N	μ	$\mu_{(dB)}$	σ^2	σ^2 (dB)
256	6.12	7.87	1.50	1.76
512	6.84	8.35	1.61	2.07
1024	7.54	8.77	1.62	2.10

dB units are $10\log_{10}(\cdot)$

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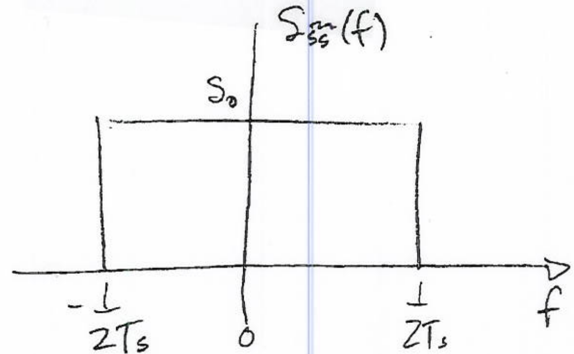

a) pdf is Rayleigh

$$\frac{1}{2}E\left[|\tilde{s}(t)|^2\right] = \frac{S_0}{T_s} \qquad E\left[|\tilde{s}(t)|^2\right] = \frac{2S_0}{T_s} = 2b_0$$

$$p_\alpha(x) = \frac{x}{b_0} e^{-x^2/2b_0} \Rightarrow b_0 = \frac{S_0}{T_s}$$

b) $P(\alpha > \theta R_{rms})$ $\qquad R_{rms} = \sqrt{2b_0}$

$= e^{-\theta^2}$

c) $$LR = \int_0^\infty \dot{\alpha}\, p(R, \dot{\alpha})\, d\dot{\alpha}$$

$$= \sqrt{\frac{1}{2\pi b_2}}\, e^{-\dot{\alpha}^2/2b_2} \cdot \frac{\alpha}{b_0} e^{-\alpha^2/2b_0}$$

$$b_n = (2\pi)^n \int_{-f_m}^{f_m} S_{ss}^c(f)\, f^n\, df$$

$$b_0 = (2\pi)^0 S_0 \int_{-1/2T_s}^{1/2T_s} df = \frac{S_0}{T_s}$$

$$b_1 = 0$$

$$b_2 = (2\pi)^2 S_0 \int_{-1/2T_s}^{1/2T_s} f^2\, df$$

$$= (2\pi)^2 S_0 \frac{f^3}{3} \Big|_{-1/2T_s}^{1/2T_s} = (2\pi)^2 \frac{2}{3} \left(\frac{1}{2T_s}\right)^3 S_0$$

$$LR = \int_0^\infty \dot{\alpha}\, p(\theta R_{rms})\, p(\dot{\alpha})\, d\dot{\alpha}$$

$$= p(\theta R_{rms}) \int_0^\infty \frac{1}{\sqrt{2\pi b_2}}\, \dot{\alpha}\, e^{-\dot{\alpha}^2/2b_2}\, d\dot{\alpha}$$

$$= \sqrt{\frac{b_2}{2\pi}}\, p(\theta R_{rms})$$

$$= \sqrt{(2\pi) \frac{2}{3} \left(\frac{1}{2T_s}\right)^3 S_0}\; p(\theta R_{rms})$$

$$= \sqrt{(2\pi) \frac{2}{3} \left(\frac{1}{2T_s}\right)^3 S_0} \cdot \sqrt{\frac{2T_s}{S_0}}\, \theta e^{-\theta^2}$$

$$= \sqrt{(2\pi) \frac{2}{3} \left(\frac{1}{2T_s}\right)^2}\, \theta e^{-\theta^2}$$

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4 / 4.22

A Guassian pulse-shaping filter has the transfer function

$$H(f) = \exp\left\{-(\frac{f}{B})^2\frac{\ln 2}{2}\right\}.$$

The impulse response of the filter is

$$\begin{aligned}
h(t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} H(f)\exp j2\pi f t df \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp\{-(\frac{f}{B})^2\frac{\ln 2}{2} + j2\pi f t\}df \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp\{-\frac{1}{B^2}\frac{\ln 2}{2}[f - j2\pi t\frac{B^2}{\ln 2}]^2\}\cdot\exp\{-2\pi^2t^2\frac{B^2}{\ln 2}\}df \\
&= \frac{1}{2\pi}\exp\{-2\pi^2t^2\frac{B^2}{\ln 2}\}\int_{-\infty}^{+\infty}\exp\{-\frac{1}{B^2}\frac{\ln 2}{2}f^2\}\cdot df \\
&= \frac{1}{\sqrt{2\pi}}\exp\{-2\pi^2t^2\frac{B^2}{\ln 2}\}\cdot B\sqrt{\frac{1}{\ln 2}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp\{-\frac{1}{2}s^2\}\cdot ds \\
&= \frac{1}{\sqrt{2\pi\ln 2}}B\exp\{-2\pi^2t^2\frac{B^2}{\ln 2}\}.
\end{aligned}$$

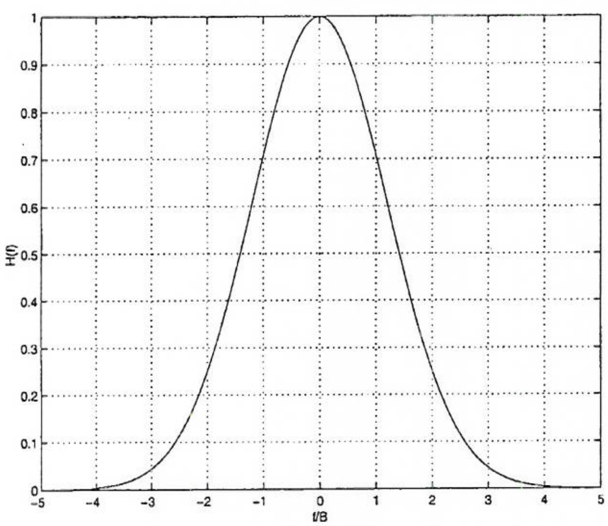
The frequence shaping pulse is

$$h_f(t) = \frac{1}{2T}\left[Q\left(\frac{t/T + 1/2}{\sigma}\right) - Q\left(\frac{t/T - 1/2}{\sigma}\right)\right],$$

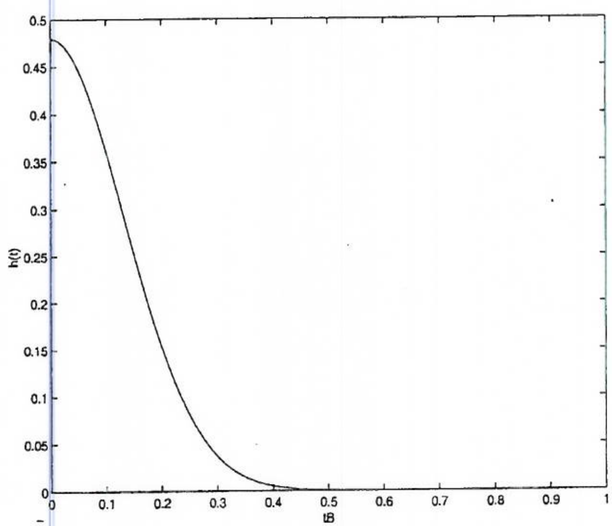
where

$$\sigma^2 = \frac{\ln 2}{4\pi^2 (BT)^2}.$$

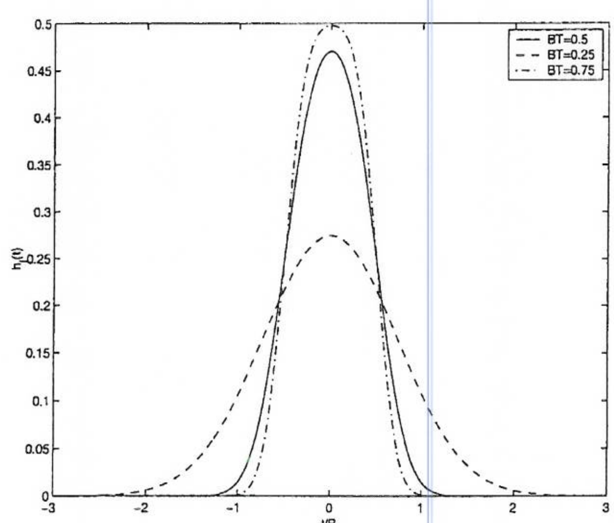
The frequency response and the impulse response of the filter, and the frequency shaping pulse $h_f(t)$ are shown in the respective figures.



Frequency Response $H(f)$



Impulse Response $h(t)$



Frequency Shaping Pulse $h_f(t)$

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28 a) Since we have an uncorrelated binary data sequence, the autocorrelation function is $\phi_{xx}(n) = \delta(n)$ and the power spectrum is $S_{xx}(f) = 1$. Following (4.196), (4.210) and (4.212), the psd of the complex envelope is

$$\begin{aligned} S_{vv}(f) &= \frac{A_m^2}{T}|H_a(f)|^2 S_{xx}(f) \\ &= \frac{A_m^2 A}{T}\tau e^{-\pi(f\tau)^2}. \end{aligned}$$

b) The peak value of the psd occurs at $f = 0$. In order to find the value of τ so that the power density spectrum is 20 dB below its peak value at frequency $1/T$, we need to solve

$$10\log_{10}\left(\frac{A_m^2 A}{T}\tau\right) - 10\log_{10}\left(\frac{A_m^2 A}{T}\tau e^{-\pi(\tau/T)^2}\right) = 20, \tag{4}$$

which is equivalent to

$$\frac{1}{e^{-\pi\frac{\tau}{T}}} = 10^2 \Rightarrow \tau = T\sqrt{\frac{2}{\pi}\ln 10} = 1.21073T. \tag{5}$$

c) In order to find the time domain pulse $h_a(t)$, we will use the fact that the Fourier transform of the Gaussian is also a Gaussian, i.e.

$$e^{-\pi t^2} \stackrel{\mathcal{F}}{\longleftrightarrow} e^{-\pi f^2}. \tag{6}$$

Then

$$H_a(f) = \sqrt{A\tau}e^{-\pi(f/(\sqrt{2}/\tau))^2} \stackrel{\mathcal{F}}{\longleftrightarrow} \sqrt{A\tau}(\sqrt{2}/\tau)e^{-\pi(\sqrt{2}/\tau t)^2} = \sqrt{\frac{2A}{\tau}}e^{-\pi 2t^2/\tau}. \tag{7}$$

6/ 4.30 a) For the 16-ary biorthogonal modulation scheme, the autocorrelation function of one biorthogonal signal $\tilde{s}_m(t)$ is

$$\begin{aligned} \phi_{\tilde{s}_m,\tilde{s}_m}(\tau) &= \frac{1}{2}\mathrm{E}[\tilde{s}_m^*(t)\tilde{s}_m(t+\tau)] \\ &= \frac{1}{2}\mathrm{E}[A^2\sum_{k=1}^{8} h_{m,k}^* h_c^*(t-kT_c)\sum_{l=1}^{8} h_{m,l}h_c(t+\tau-lT_c)] \\ &= \frac{A^2}{2}\sum_{k=1}^{8}\sum_{l=1}^{8} h_{m,k}^* h_{m,l}\mathrm{E}[h_c^*(t-kT_c)h_c(t+\tau-lT_c)]. \end{aligned}$$

Using time autocorrelation for signal $\tilde{s}_m(t)$, define time autocorrelation function of $h_c(t)$ as,

$$\phi_{h_c}(\tau) = \int_{-\infty}^{+\infty} h_c^*(t)h_c(t+\tau)dt.$$

Set $l - k = n$. Since $h_c(t) \neq 0$ only for $0 \le t \le T_c$, we have

$$\begin{aligned} \phi_{\tilde{s}_m,\tilde{s}_m}(\tau) &= \frac{A^2}{2}\sum_{k=1}^{8}\sum_{l=1}^{8} h_{m,k}^* h_{m,l}\phi_{h_c}(\tau-(l-k)T_c) \\ &= \begin{cases} \frac{A^2}{2}\sum_{n=0}^{7}\sum_{k=1}^{8-n} h_{m,k}^* h_{m,k+n}\phi_{h_c}(\tau-nT_c) & n \ge 0 \\ \frac{A^2}{2}\sum_{n=-7}^{-1}\sum_{k=1}^{8+n} h_{m,k}^* h_{m,k-n}\phi_{h_c}(\tau-nT_c) & n < 0 \end{cases} \\ &= \frac{A^2}{2}\sum_{n=-7}^{7}\phi_{mm}^a(n)\phi_{h_c}(\tau-nT_c), \end{aligned}$$

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where $\phi^a_{mm}(n)$ is the aperiodic autocorrelation function of sequence $h_{m,k}$. The Fourier transform of $\phi_{h_c}(\tau)$ is

$$\phi_{h_c}(\tau) \Longleftrightarrow |H_c(f)|^2.$$

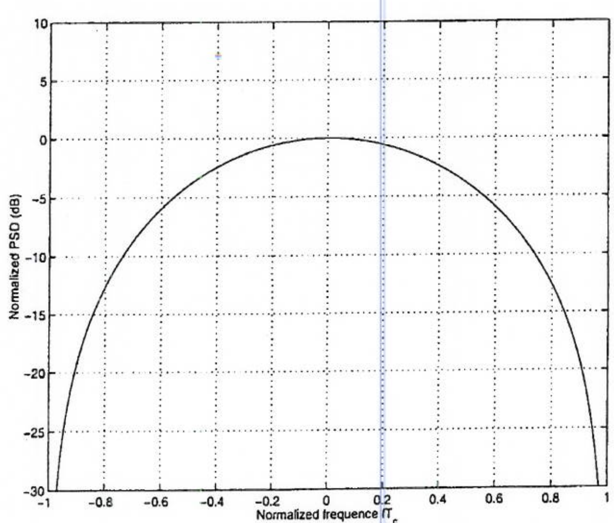
For $h_c(t) = u_{T_c}(t)$, the power spectrum density function is

$$\begin{aligned} S_{\tilde{s}_m\tilde{s}_m}(f) &= \frac{A^2}{2}|H_c(f)|^2 \sum_{n=-7}^{7} \phi^a_{mm}(n)e^{-j2\pi f n T_c} \\ &= \frac{(AT_c)^2}{2}\mathrm{sinc}^2(fT_c)\Phi_{mm}(f). \end{aligned}$$

There are 16 signals. We can take the average for the psd. Therefore,

$$\begin{aligned} S_{\tilde{s}\tilde{s}}(f) &= \frac{1}{16}\sum_{m=1}^{16} S_{\tilde{s}_m\tilde{s}_m}(f) \\ &= \frac{(AT_c)^2}{32}\sum_{m=1}^{16} \mathrm{sinc}^2(fT_c)\Phi_{mm}(f) \end{aligned}$$

b) The psd is shown here.



The power spectrum $S_{\tilde{s}\tilde{s}}(f)$

Note: $T_c = T_b/2$