ECE 6604 Assignment #6 Solutions

1) 5.6 a) $$p_{\alpha_c}(x) = \int_0^\infty p_{\alpha|\Omega_p}(x|\omega)\, p_{\Omega_p}(\omega)\, d\omega$$

For the case of Rayleigh fading

$$\Omega_p = E[\alpha^2(t)] = 2b_0$$

$$p_{\alpha|\Omega_p}(x|\omega) = \frac{2x}{\omega} e^{-x^2/\omega}$$

$$p_{\alpha_c}(x) = \int_0^\infty \frac{2x}{\omega} e^{-x^2/\omega} \frac{1}{w\sigma_\Omega \xi \sqrt{2\pi}} e^{-\frac{(10\log_{10} w - \mu_{\Omega_p (dBm)})^2}{2\sigma_\Omega^2}} dw$$

b) We have

$$P_{\alpha_c^2}(x) = \int_0^\infty \frac{1}{w} e^{-x/w} \frac{1}{w\sigma_\Omega \xi \sqrt{2\pi}} e^{-\frac{(10\log_{10}(w/\mu_{\Omega_p}))^2}{2\sigma_\Omega^2}} dw$$

where $\mu_{\Omega_p} = 10^{(\mu_{\Omega_p}(dBm) - 30)/10}$ watts

Let $y = w/\mu_{\Omega_p}$ $\quad w = \mu_{\Omega_p} y$ $\quad dw = \mu_{\Omega_p} dy$

$$P_{\alpha_c^2}(x) = \frac{1}{\sigma_\Omega \xi \sqrt{2\pi}} \int_0^\infty \frac{1}{\mu_{\Omega_p} y^2} e^{-\frac{x}{\mu_{\Omega_p} y}} \cdot e^{-\frac{1}{2\sigma_\Omega^2}[10\log_{10} y]^2} dy$$

For BPSK

$$P_b = Q(\sqrt{2\gamma_b})$$

where

$$\gamma_b = \frac{\alpha_c^2}{N_o}$$

is the "instantaneous" bit energy-to-noise ratio.

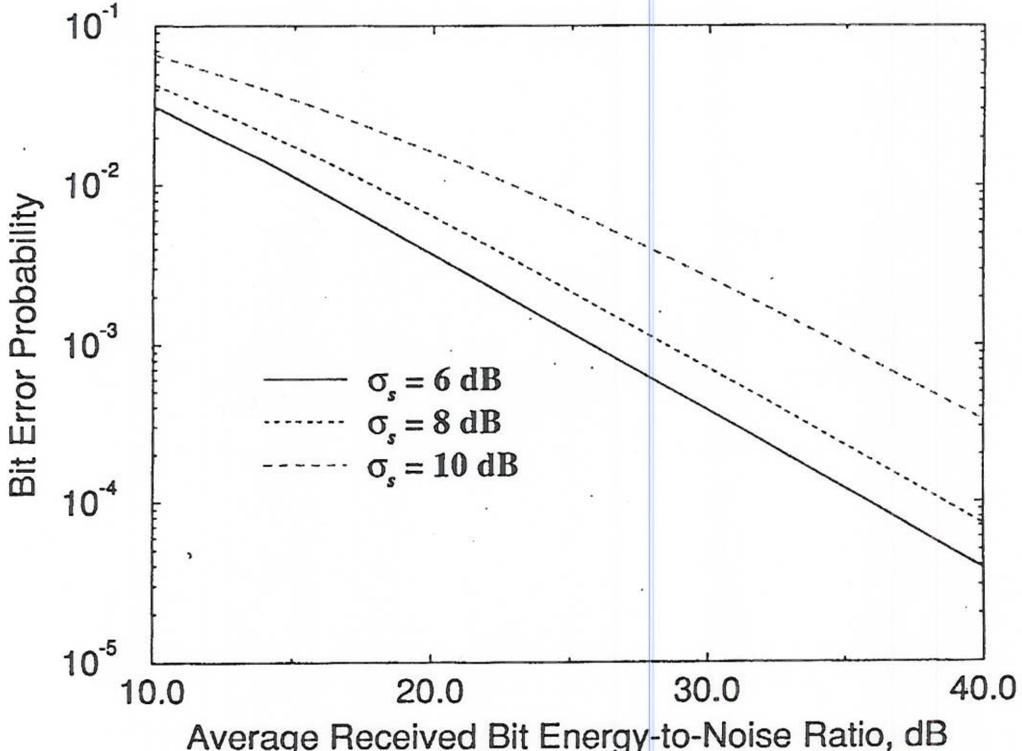
$$P_b = \int_0^\infty Q(\sqrt{2x/N_o})\, P_{\alpha_c^2}(x)\, dx$$

$$= \frac{1}{\sigma_\Omega \xi \sqrt{2\pi}} \int_0^\infty \Bigg\{ \underbrace{\int_0^\infty Q\left(\sqrt{\frac{2x}{N_o}}\right) \frac{1}{\mu_{\Omega_p} y} e^{-\frac{x}{\mu_{\Omega_p} y}} dx}_{= \frac{1}{2}\left[1 - \sqrt{\frac{\bar{\gamma} y}{1+\bar{\gamma} y}}\right]} \Bigg\} \frac{1}{y} e^{-\frac{1}{2\sigma_\Omega^2}[10\log_{10} y]^2} dy$$

$$, \quad \bar{\gamma} = \frac{\mu_{\Omega_p}}{2N_o}$$

$$\Rightarrow P_b = \frac{1}{\sigma_\Omega \, \xi \sqrt{2\pi}} \int_0^\infty \frac{1}{2}\left[1 - \sqrt{\frac{\bar{\gamma} y}{1+\bar{\gamma} y}}\,\right] \frac{1}{y} e^{-\frac{1}{2\sigma_\Omega^2}\left[10\log_{10} y\right]^2} dy$$

A plot of P_b vs. $\bar{\gamma}$ for various σ_Ω is shown.



2)

4

5.9 (a)

$$
\begin{aligned}
R_k &= \tilde{s}(kT_s + \Delta_t) \\
&= A \sum_{n=0}^{N-1} x_n e^{\frac{j2\pi kn}{N}} e^{\frac{j2\pi n\Delta_t}{NT_s}}.
\end{aligned}
$$

Taking FFT on the received samples,

$$
\begin{aligned}
Z_i &= \frac{1}{N} \sum_{k=0}^{N-1} R_k e^{-\frac{j2\pi ki}{N}} \\
&= \frac{A}{N} \sum_{k=0}^{N-1} \sum_{n=0}^{N-1} x_n e^{\frac{j2\pi kn}{N}} e^{\frac{j2\pi n\Delta_t}{NT_s}} e^{-\frac{j2\pi ki}{N}} \\
&= \frac{A}{N} \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} x_n e^{\frac{j2\pi k(n-i)}{N}} e^{\frac{j2\pi n\Delta_t}{NT_s}} \\
&= A x_i e^{\frac{j2\pi n\Delta_t}{NT_s}}.
\end{aligned}
$$

Timing offset smaller than the guard interval results in a phase shift.

(b) Let us assume that M out of N samples come from different OFDM block due to timing offset. Then,

$$
R_k = \begin{cases} A \sum_{n=0}^{N-1} x_n e^{\frac{j2\pi kn}{N}} e^{\frac{j2\pi n\Delta_t}{NT_s}} & (0 \le k \le N-M-1) \\ A \sum_{n=0}^{N-1} x'_n e^{\frac{j2\pi kn}{N}} e^{\frac{j2\pi n\Delta_t}{NT_s}} & (N-M \le k \le N-1) \end{cases}
$$

Therefore,

$$
\begin{aligned}
Z_i &= \frac{1}{N} \sum_{k=0}^{N-1} R_k e^{-\frac{j2\pi ki}{N}} \\
&= \frac{A}{N} \sum_{n=0}^{N-1} \sum_{k=0}^{N-M-1} x_n e^{\frac{j2\pi k(n-i)}{N}} e^{\frac{j2\pi n\Delta_t}{NT_s}} + \frac{A}{N} \sum_{n=0}^{N-1} \sum_{k=N-M}^{N-1} x'_n e^{\frac{j2\pi k(n-i)}{N}} e^{\frac{j2\pi n\Delta_t}{NT_s}} \\
&= A x_i e^{\frac{j2\pi n\Delta_t}{NT_s}} + \frac{A}{N} \sum_{n=0}^{N-1} \sum_{k=N-M}^{N-1} (x'_n - x_n) e^{\frac{j2\pi k(n-i)}{N}} e^{\frac{j2\pi n\Delta_t}{NT_s}}.
\end{aligned}
$$

Timing offset greater than the guard channel introduces ISI.

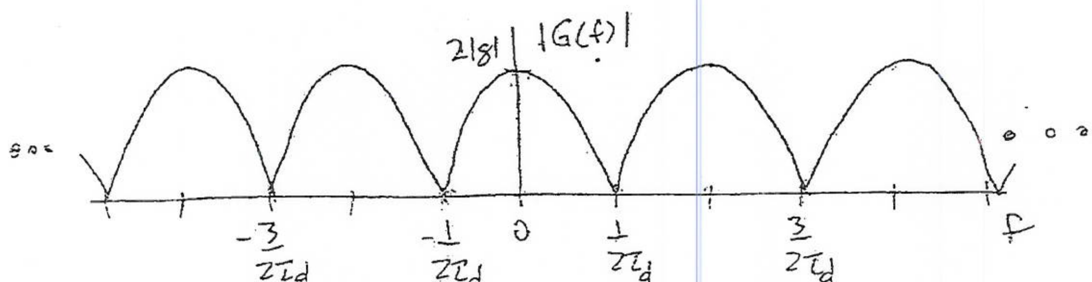
3)

10.1 The sub-carriers of the OFDM complex envelope (assuming the complex envelope is centered at 0 Hz) are placed at 2000k Hz, $k = \pm 1, \pm 3, \pm 5, \ldots$

The channel transfer function is

$$G(f) = \mathcal{F}\{g(t,\tau)\}$$
$$= g\left(1 + e^{-j2\pi f \tau_d}\right)$$
$$= 2g e^{-j\pi f \tau_d}\left(\frac{e^{j\pi f \tau_d} + e^{-j\pi f \tau_d}}{2}\right)$$
$$= 2g e^{-j\pi f \tau_d} \cos(\pi f \tau_d)$$

$$|G(f)| = 2|g|\,|\cos(\pi f \tau_d)|$$



Now choose τ_d so the subcarriers are on the nulls of $|G(f)|$

$$\frac{k}{2\tau_d} = 2000k, \quad k \text{ odd}$$

$$\text{or } \tau_d = .25 \times 10^{-3}$$

4)

6.1 $\text{rms} = \sqrt{E[X_n^2]} = \frac{1}{\sqrt{N}} \implies E[X_n^2] = \frac{1}{N}$

Since the X_i are Rayleigh distributed, the X_i^2 are exponentially distributed

$$P_{X_i^2}(x) = \frac{1}{2\sigma^2} e^{-x/2\sigma^2}, \quad x \geq 0 \qquad 2\sigma^2 = E[X_n^2] = 1/N$$

$$= N e^{-xN}, \quad x \geq 0$$

a) $Y = \max(X_1^2, X_2^2, \ldots, X_N^2)$

$$F_Y(y) = [F_{X_i^2}(y)]^N = \left[\int_0^y N e^{-xN}\,dx\right]^N = [1 - e^{-yN}]^N$$

$$P_Y(y) = \frac{d}{dy} F_Y(y) = N^2 [1 - e^{-yN}]^{N-1} e^{-yN}, \quad y \geq 0$$

b) $Y = X_1^2 + X_2^2 + \cdots + X_N^2$

The X_i^2 are exponentially distributed or central chi-square distributed with 2 degrees of freedom. The sum of N i.i.d. central chi-square random variables with 2 degrees of freedom is another central chi-square random variable with 2N degrees of freedom (see Appendix A.35)

$$P_Y(y) = \frac{1}{(2\sigma^2)^N (N-1)!} y^{N-1} e^{-y^2/2\sigma^2}, \quad y \geq 0$$

$$= \frac{N^N}{(N-1)!} y^{N-1} e^{-yN}, \quad y \geq 0.$$

5)

6.2 $P_{\gamma_1}(x) = \frac{1}{\bar\gamma_1} e^{-\frac{x}{\bar\gamma_1}}$ and $P_{\gamma_2}(x) = \frac{1}{\bar\gamma_2} e^{-\frac{x}{\bar\gamma_2}}$

Predetection selective combining $\implies \gamma_b^s = \max\{\gamma_1, \gamma_2\} \iff F_{\gamma_b^s}(x) = P_r[(\gamma_1 \leq x) \cap (\gamma_2 \leq x)]$

γ_1 and γ_2 are independent $\implies F_{\gamma_b^s}(x) = (1 - e^{-\frac{x}{\bar\gamma_1}})(1 - e^{-\frac{x}{\bar\gamma_2}})$

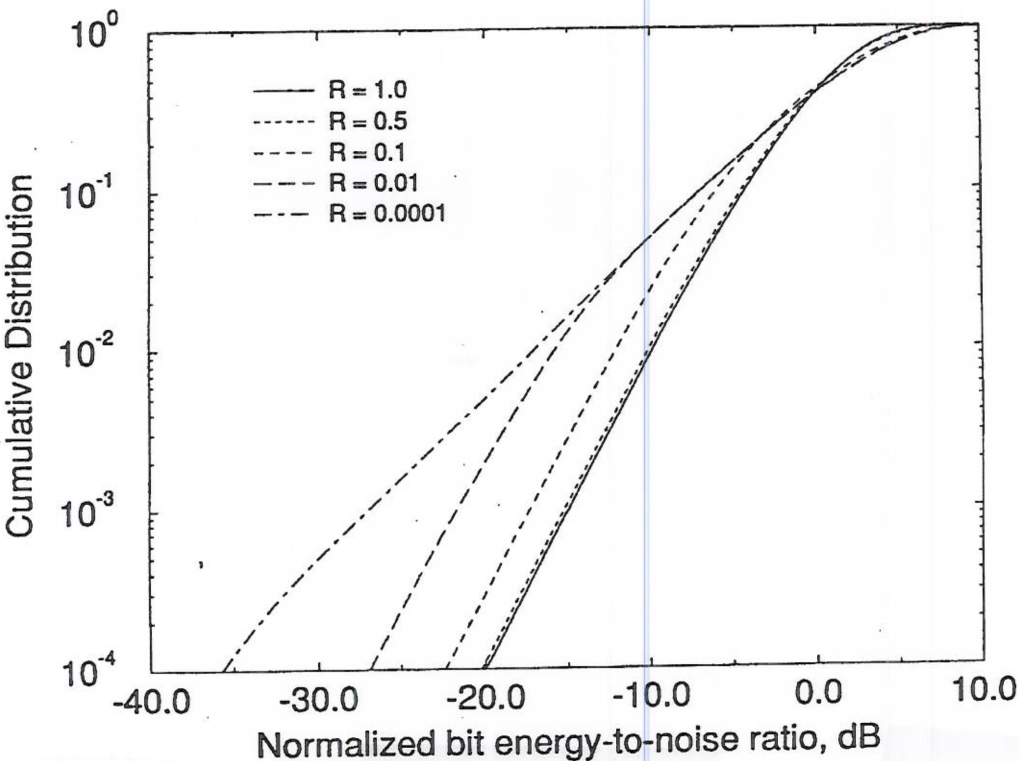
$\bar\gamma_t = \frac{\bar\gamma_1 + \bar\gamma_2}{2}$, $\xi = \frac{\bar\gamma_1}{\bar\gamma_2} \implies \bar\gamma_1 = \frac{2\xi}{1+\xi}\bar\gamma_t$ and $\bar\gamma_2 = \frac{2}{1+\xi}\bar\gamma_t$

thus $F_{\gamma_b^s}(x) = \left\{1 - e^{-\frac{1+\xi}{2\xi\bar\gamma_t}x}\right\} \cdot \left\{1 - e^{-\frac{1+\xi}{2\bar\gamma_t}x}\right\}$

A plot of $F_{\gamma_b^s}(x)$ against $10\log_{10}(\frac{x}{\bar\gamma_t})$ is shown below; the parameter R is the symbol ξ.

Note that when $F_{\gamma_b^s}(x) = 10^{-4}$, then $10\log_{10}(\frac{x}{\bar\gamma_t}) \approx -37$ dB. In Figure 5.14, $F_{\gamma_b^s}(x) = 10^{-4}$ at -40 dB. The factor of 3dB arises because we are plotting $\bar\gamma_t/2$.

Finally, note that when $\xi = 0.5$, we still get most of the diversity gain.



6) 6.9 $\quad \gamma_s^{mr} = \sum_{k=1}^{2} \gamma_k = \sum_{k=1}^{2} \alpha_k^2 \frac{E_b}{N_0}$

and $\bar{\gamma}_c = E[\gamma_k]$

But $\quad P_{\alpha_k}(x) = 0.9\,\delta(x-1.0) + 0.1\,\delta(x-0.05)$

$$\bar{\gamma}_c = 0.9\frac{E_b}{N_0} + 0.1\,(0.05)^2\frac{E_b}{N_0}$$

$$= 0.90025\frac{E_b}{N_0}$$

and

$$p_{\gamma_s^{mr}}(x) = (0.9)^2 \delta\left(x - (1+1)E_b/N_o\right)$$
$$+ 2(0.9)(0.1)\,\delta\left(x - (1+0.05^2)E_b/N_o\right)$$
$$+ (0.1)^2 \delta\left(x - (0.05^2+0.05^2)E_b/N_o\right)$$
$$= .81\,\delta(x - 2E_b/N_o)$$
$$+ .18\,\delta(x - 1.0025\,E_b/N_o)$$
$$+ .01\,\delta(x - 0.005\,E_b/N_o)$$

For BPSK

$$P_b(\gamma_s^{mr}) = Q\left(\sqrt{2\gamma_s^{mr}}\right)$$

$$P_b = \int_0^\infty P_b(x)\, p_{\gamma_s^{mr}}(x)\,dx$$

$$= \int_0^\infty Q(\sqrt{2x})\left[0.81\,\delta(x - 2E_b/N_o) + .18\,\delta(x - 1.0025\,E_b/N_o) + 0.01\,\delta(x - 0.005\,E_b/N_o)\right]$$

$$= 0.81\,Q\left(\sqrt{4\,E_b/N_o}\right)$$
$$+ 0.18\,Q\left(\sqrt{2.005\,E_b/N_o}\right)$$
$$+ 0.01\,Q\left(\sqrt{0.01\,E_b/N_o}\right)$$

But $\frac{E_b}{N_o} = 1.110803\,F_c$

$$\Rightarrow P_b = 0.81\, Q\left(\sqrt{4.443210\,\bar{\gamma}_c}\right) + 0.18\, Q\left(\sqrt{2.227159\,\bar{\gamma}_c}\right) + 0.01\, Q\left(\sqrt{0.011108\,\bar{\gamma}_c}\right)$$